# Undecidability as a genuine quantum property

Jens Eisert[1], Markus P. Müller[2], and Christian Gogolin[1]

[1]Dahlem Center for Complex Quantum Systems, Freie Universität Berlin

[2]Perimeter Institute for Theoretical Physics, Waterloo, Canada

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UNAM Mexico

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- What it means:
  - Statement about the complexity of the mathematical theory of quantum mechanics
  - Arguably the strongest complexity theoretic quantum/classical separation imaginable (almost)

# Outline

1 Undecidability
- Turing Machines and Computability
- Undecidability and the halting problem

2 Measurement occurrence problem
- Operational definition
- Quantum and classical version

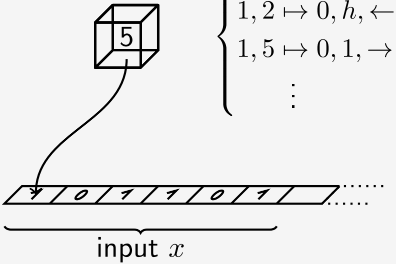
3 Results
- Decidability of the classical problem
- Undecidability of the quantum problem

# Undecidability

# Turing machines

Turing Machine $\sharp t$

$$\begin{cases} 0,1 \mapsto 0,2,\rightarrow \\ 1,2 \mapsto 0,h,\leftarrow \\ 1,5 \mapsto 0,1,\rightarrow \\ \quad\vdots \end{cases}$$



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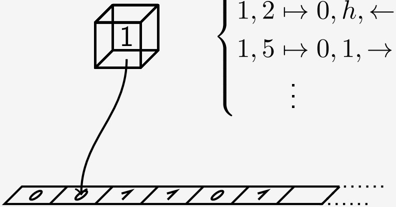

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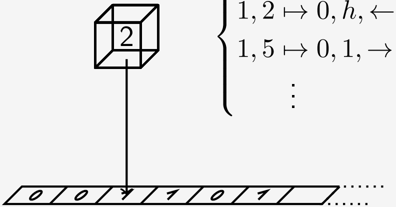

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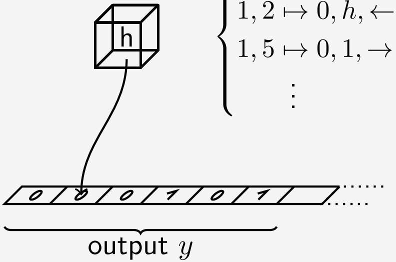

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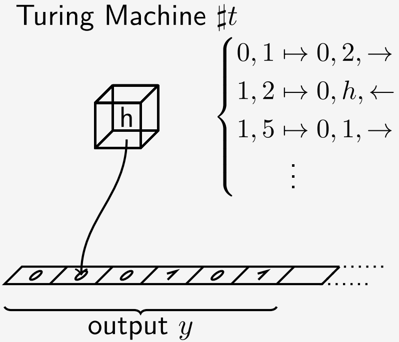


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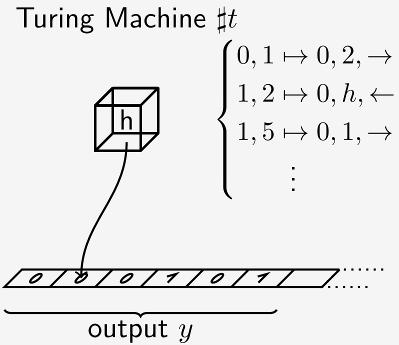


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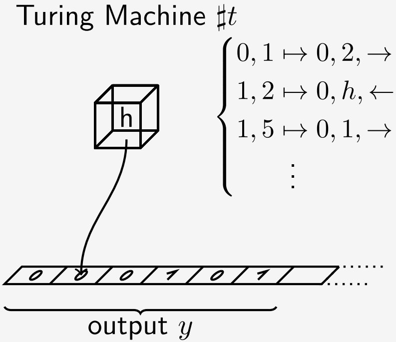


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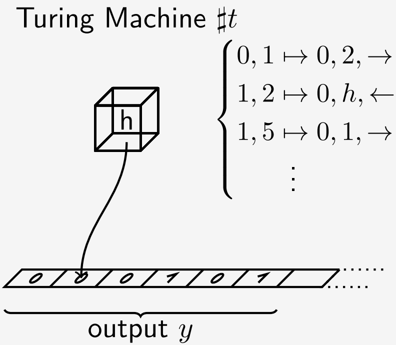


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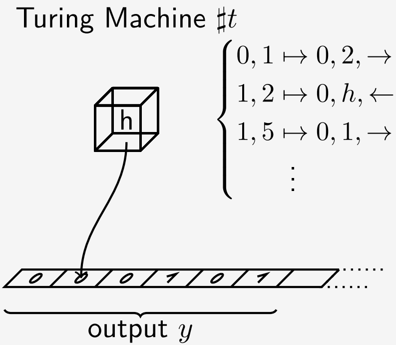


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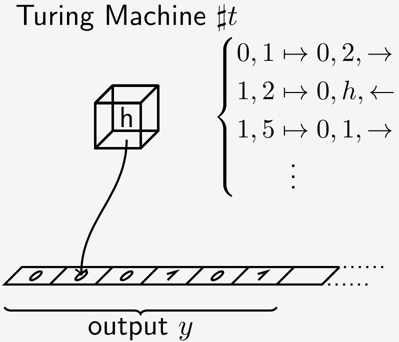


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- Universal model of computation

- Question: Are all functions $f : \mathbb{N} \rightarrow \{0, 1\}$ computable?
- Answer: No!

# The halting problem

$\mathrm{Halt}(x)$

$$\mathrm{Halt}(x) = \begin{cases} \text{true} & \text{if TM } \sharp x \text{ halts on input } x \\ \text{false} & \text{if TM } \sharp x \text{ does not halt on input } x \end{cases}$$

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if Halt(x) then
    loop forever
else
    halt
endif
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What is $f(\sharp f)$?

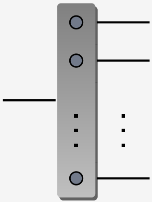
$\lightning$

### Turing undecidability

A problem is undecidable iff there is no algorithm solving each instance of the problem.

# Measurement occurrence problem

# A problem from measurement theory



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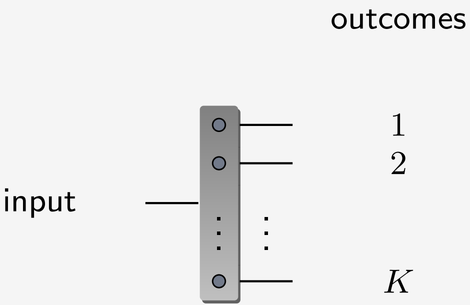

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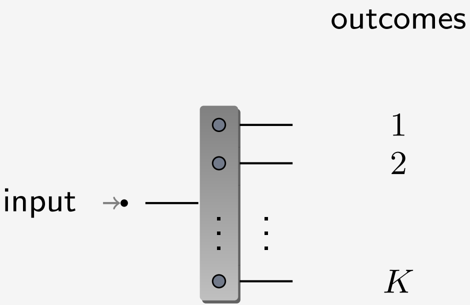

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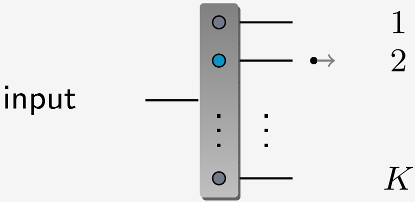

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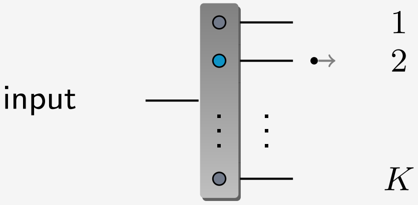


$$w_1 = 2$$

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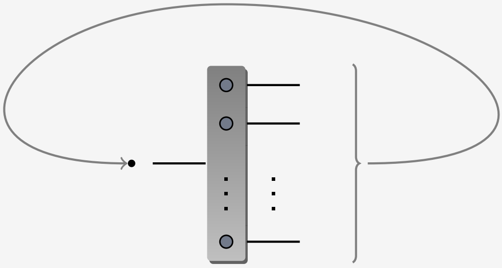

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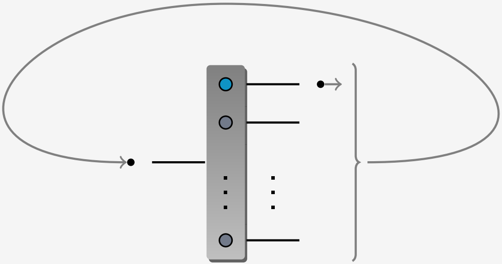

$$w_2 = 1$$

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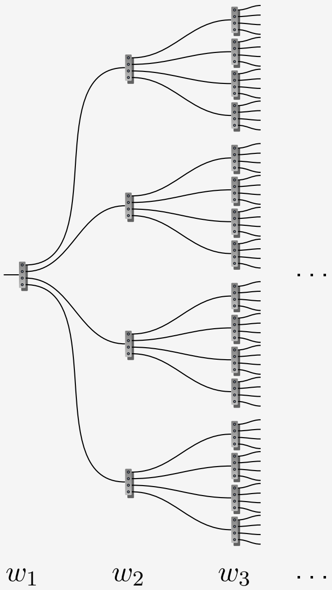

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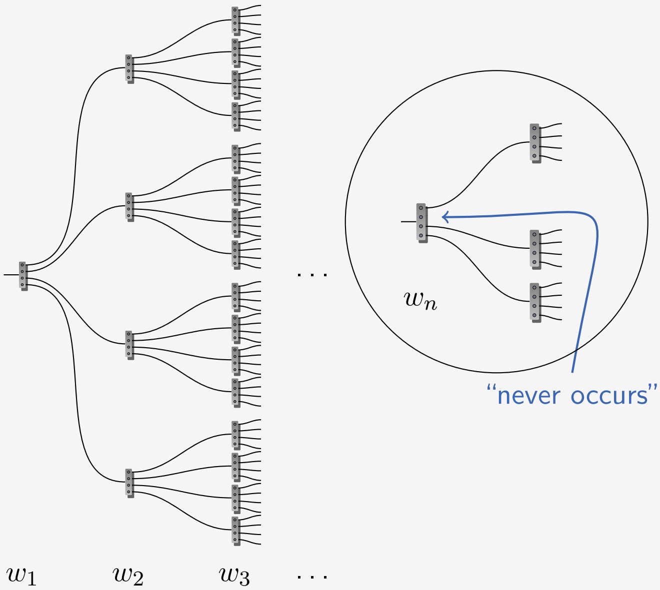

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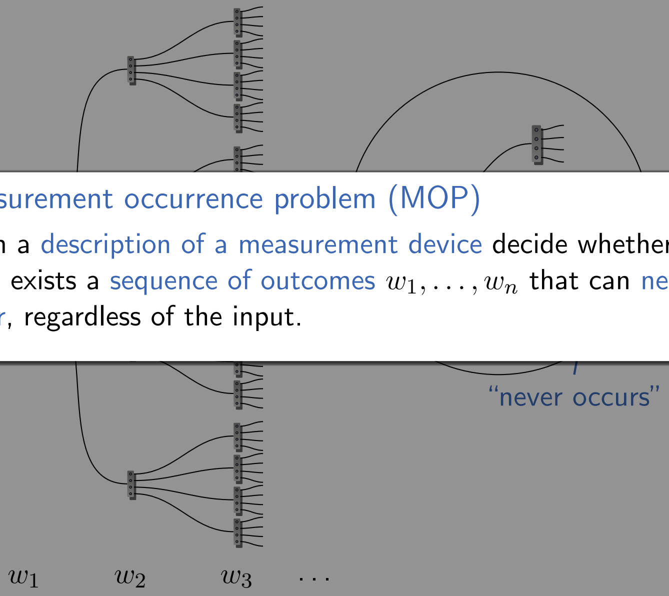


## Measurement occurrence problem (MOP)

Given a description of a measurement device decide whether there exists a sequence of outcomes $w_1, \ldots, w_n$ that can never occur, regardless of the input.

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| | $\sum_{j=1}^{K} A_j^\dagger A_j = 1$ | $\sum_{j=1}^{K} Q_j$ is stochastic |
| on outcome $j$: | $\rho \mapsto \dfrac{A_j \rho A_j^\dagger}{\operatorname{Tr}[A_j \rho A_j^\dagger]}$ | $\vec{p} \mapsto \dfrac{Q_j \vec{p}}{\sum_{i=1}^{d} (Q_j \vec{p})_i}$ |
| $\operatorname{Prob}(w_1, \dots, w_n) =$ | $\operatorname{Tr}[A_{w_1}^\dagger \dots A_{w_n}^\dagger A_{w_n} \dots A_{w_1} \rho]$ | $\sum_{i=1}^{d} (Q_{w_n} \dots Q_{w_1} \vec{p})_i$ |

# When is $w_1, \ldots w_n$ an impossible outcome sequence?

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- Remember: different restrictions on $A_j$ and $Q_j$!

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# Results

# Quantum vs. classical separation

**Theorem 1 (Undecidability)**

*The quantum measurement occurrence problem (QMOP) for $K \geq 9$ and $d \geq 15$ is undecidable.*

**Theorem 2 (Decidability)**

*For any $K$ and $d$, both QMOP with Kraus operators $A_j$ with non-negative entries and CMOP are decidable.*

[1] J. Eisert, M. P. Mueller, and C. Gogolin, Phys. Rev. Lett. 108, 260501 (2012)

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- Ergo: Check all words $M'_{w_n} * \dots * M'_{w_1}$ of length $\leq 2^{(d^2)}$.

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**Matrix mortality problem (MMP) is undecidable [2, 3]**

It is undecidable whether the semi-group generated by $\{M_i\}_{i=1}^{K} \subset \mathbb{Z}_{d\times d}$ (with $K \geq 8$ and $d \geq 3$) contains the zero matrix.

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- Find clever encoding of the MMP into the QMOP such that

$$\exists w_n, \ldots, w_1 : \; A_{w_n} \ldots A_{w_1} = 0 \iff \exists i_{n'}, \ldots, i_1 : \; M_{i_n} \ldots M_{i_{n'}} = 0.$$

and which takes the restrictions on the $A_j$ into account.

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- What it means:
  - Destructive interference makes the quantum problem undecidable.
  - Statement about the complexity of the mathematical theory of quantum mechanics
  - Arguably the strongest complexity theoretic quantum/classical separation imaginable (almost: CMOP is NP-complete)

# References

## Thank you for your attention!

⟶ slides: www.cgogolin.de

[1] J. Eisert, M. Müller, and C Gogolin.
Quantum measurement occurrence is undecidable.
*Physical Review Letters*, 108(26):1–5, June 2012.

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Undecidability Bounds for Integer Matrices using Claus Instances.
Technical Report 766, TUCS Turku Center for Computer Science, April 2006.

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# Making the result more physical

The QMOP asks

$$\exists w_1, \dots, w_n : \quad \mathrm{Prob}(w_1, \dots, w_n) = 0.$$

The undecidability of the QMOP is stable in the sens that the question

$$\exists w_1, \dots, w_n : \quad \mathrm{Prob}(w_1, \dots, w_n) < \delta^{-n}$$

is still undecidable, where $\delta > 1$ is a simple function of $\rho$ and the Kraus Operators $\{A_j\}_{j=1}^{K}$.