Under what conditions do quantum systems thermalize?

Christian Gogolin

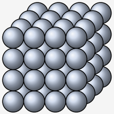
Dahlem Center for Complex Quantum Systems, Freie Universität Berlin

2014-01-20 COST conference

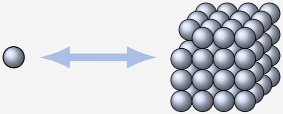
Old questions and new results

How do quantum mechanics and statistical mechanics go together?

Understanding thermalization



Understanding thermalization



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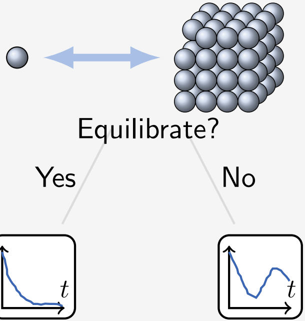

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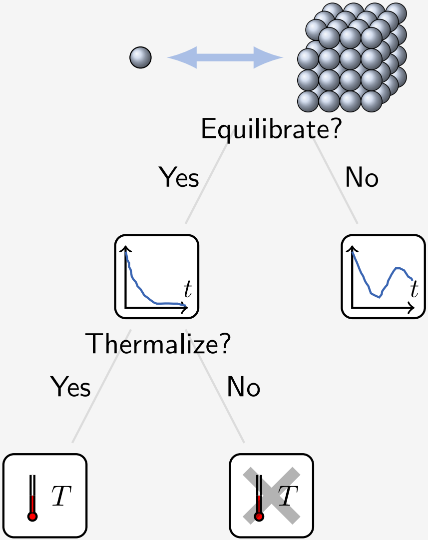

Setup

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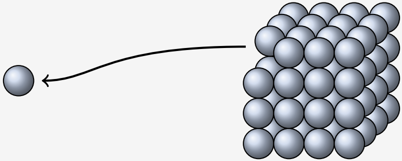
Subsystem, $\mathcal{H}_S, \mathscr{H}_S$

$d_S = \dim(\mathcal{H}_S)$

Bath, $\mathcal{H}_B, \mathscr{H}_B$

$d_B \gg d_S$



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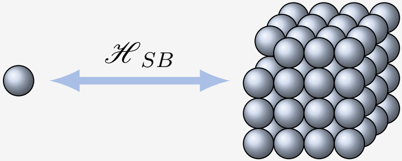

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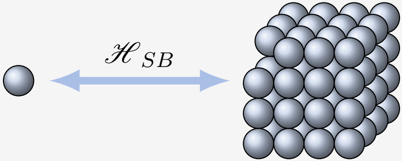

$$\psi_t^S = \mathrm{Tr}_B[\psi_t]$$

$$\mathscr{H} = \mathscr{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_B + \mathscr{H}_{SB} \qquad \frac{d\psi_t}{dt} = \mathrm{i}\,[\psi_t, \mathscr{H}]$$

Equilibration

Equilibration

Theorem 1 (Equilibration on average [3])

If $\mathscr{H}$ has non-degenerate energy gaps, then for every $\psi_0 = |\psi_0\rangle\langle\psi_0|$ there exists a ω^S such that:

$$\overline{\mathcal{D}(\psi_t^S, \omega^S)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}.$$

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[2] P. Reimann, PRL 101, 190403 (2008).
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[5] P. Reimann and M. Kastner, NJP 14, 043020 (2012).

Equilibration

Theorem

If $\mathscr{H}$ *has* … $\langle\psi_0|$

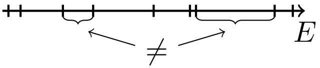
there exis …

Non-degenerate energy gaps

$\mathscr{H}$ has non–degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\Longrightarrow k = l \wedge m = n \quad \vee \quad k = m \wedge l = n$$



Intuition: Sufficient for $\mathscr{H}$ to be fully interactive

$$\mathscr{H} \neq \mathscr{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_2$$

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Effective dimension

$$d^{\text{eff}} = \frac{1}{\sum_k |\langle E_k|\psi_0\rangle|^4}.$$

Intuition: Dimension of supporting energy subspace

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$$\overline{\mathcal{D}(\psi_t^S, \omega^S)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}.$$

$\Longrightarrow$ If $d^{\text{eff}} \gg d_S^2$ then ψ_t^S equilibrates on average.

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Maximum entropy principle

Theorem 2 (Maximum entropy principle [6])

If $\mathrm{Tr}[A\,\psi_t]$ *equilibrates on average, it equilibrates towards its time average*

$$\overline{\mathrm{Tr}(A\,\psi_t)} = \mathrm{Tr}(A\,\overline{\psi_t}) = \mathrm{Tr}(A\,\omega),$$

where

$$\omega := \sum_k \Pi_k\,\psi_0\,\Pi_k$$

is the dephased state that maximizes the von Neumann entropy, given all conserved quantities (Π_k *are the energy eigen projectors).*

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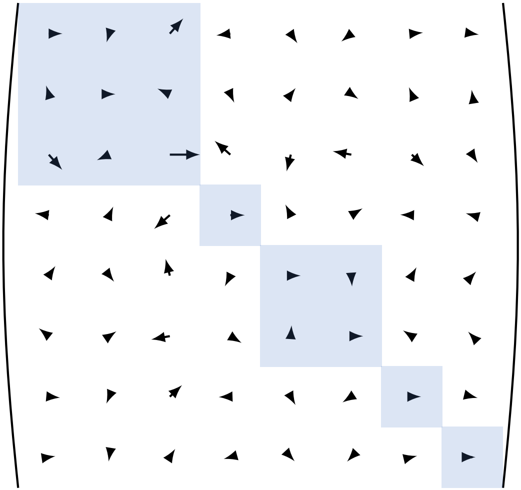
Maximu

Time averaging

$\psi_0 =$



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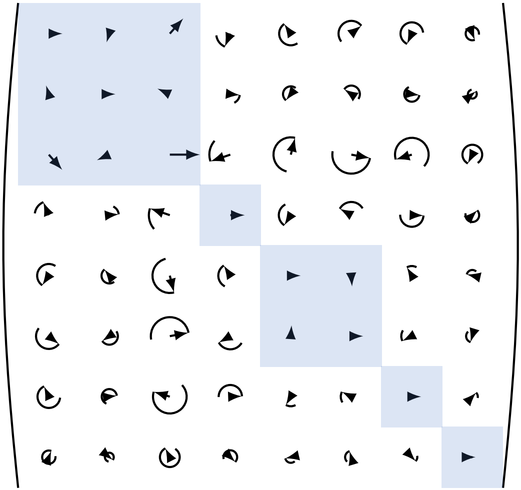

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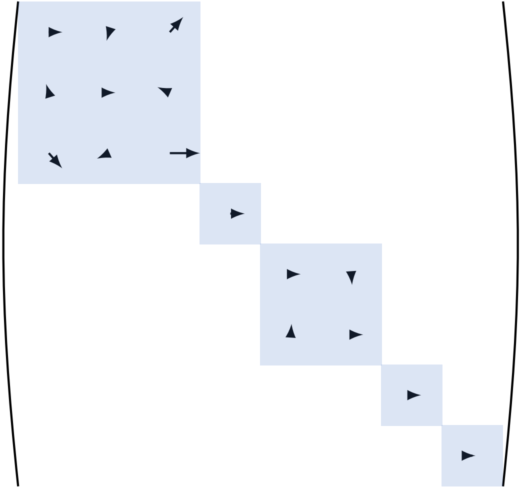

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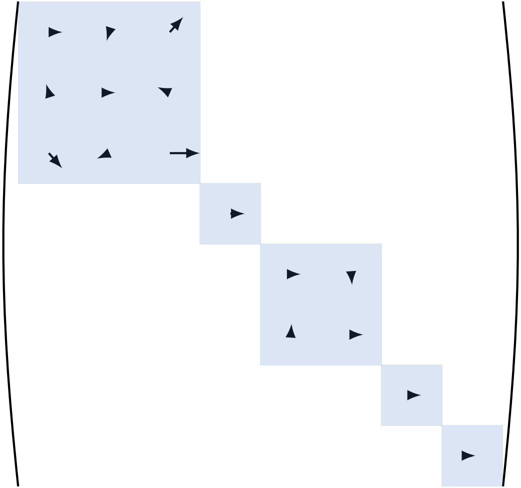
$\omega =$ 

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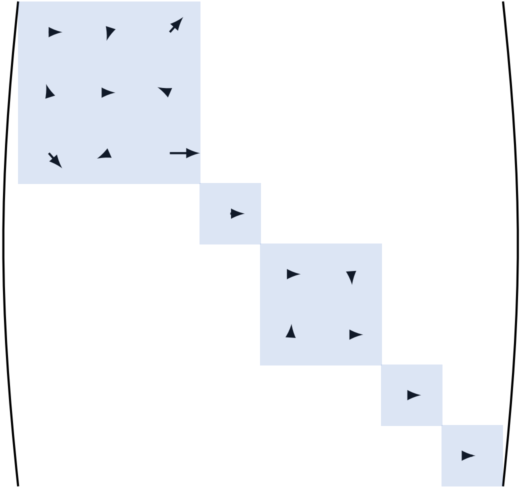
$\omega =$ 

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$\Rightarrow$ Maximum entropy principle from pure quantum dynamics.

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all con

Interesting open questions:

- Do we really need all (exponentially many) conserved quantities?
- If not, then which?
- Does this depend on integrability of the model?
- What is the relation to the GGE?

$\Rightarrow$ Maximum entropy principle from pure quantum dynamics.

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Thermalization

Thermalization is a complicated process

Thermalization implies:

1. Equilibration [1, 2, 3, 7]
2. Subsystem initial state independence [6, 8]
3. Weak bath state dependence [9]
4. Diagonal form of the subsystem equilibrium state [10]
5. Gibbs state $\omega^S = \mathrm{Tr}_B(\omega) \approx \mathrm{e}^{-\beta \mathscr{H}_S}$ [9]

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[7] J. Gemmer, M. Michel, and G. Mahler, Springer (2009).
[8] A. Hutter and S. Wehner, PRA 87, 012121 (2013).
[9] A. Riera, C. Gogolin, and J. Eisert, PRL 108, 080402 (2012).
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Thermalization and quantum integrability

There is a common belief in the literature [11, 12, 13, 14, 15] . . .

Non-integrable	$\Longrightarrow$	Thermalization
Integrable	$\Longrightarrow$	No thermalization

[11] C. Kollath et. al PRL 98, 180601 (2007).
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[13] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
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Non-integrable	$\Longrightarrow$	Thermalization
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Notions of (non-)integrability

A system is with n degrees of freedom is integrable if

- there exist n (local) conserved mutually commuting linearly/algebraically independent operators.
- the system is integrable by the Bethe ansatz.
- the system exhibits nondiffractive scattering.
- ...
- the quantum many-body system is exactly solvable in some way.

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And non-integrable otherwise?

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Lack of imagination?

See also [16]: J.-S. Caux and J. Mossel, JSM 2011, 02 (2011).

Absence of thermalization in non integrable systems

Result (Theorem 1 and 2 in [6]):

- Too little (geometric) entanglement in the energy eigenbasis prevents subsystem initial state independence.
- This can happen even in non-integrable systems.

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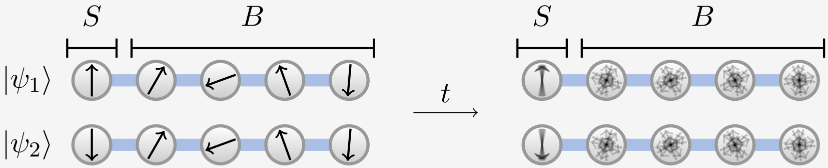

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Absence of thermalization in non integrable systems

The model:

Spin-$1/2$ XYZ chain with random coupling and on-site field.

$$\mathscr{H} = \sum_{i=1}^{n} h_i \, \sigma_i^Z + \sum_{i=1}^{n-1} \vec{b}_i \cdot \vec{\sigma}_i^{\mathrm{NN}}$$

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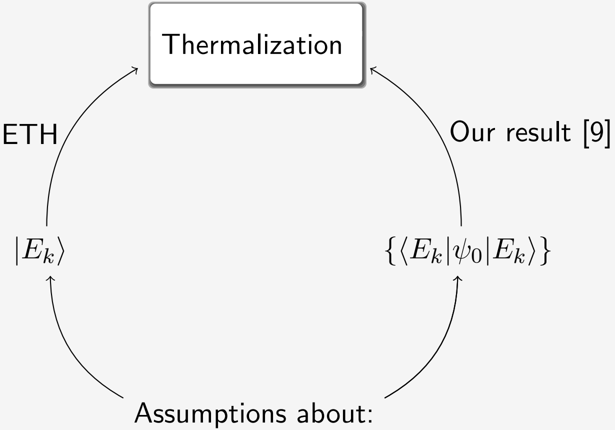
Interesting open questions:

- What is the relation to Anderson localization?
- Can this also happen in translation invariant systems?
- Why is integrability a useful concept despite the ambiguity?

[6] C. Gogolin, M. P. Müller, and J. Eisert, PRL 106, 040401 (2011).

Conditions for thermalization

Two ways to prove thermalization



[9] A. Riera, C. Gogolin, and J. Eisert, PRL 108, 080402 (2012).
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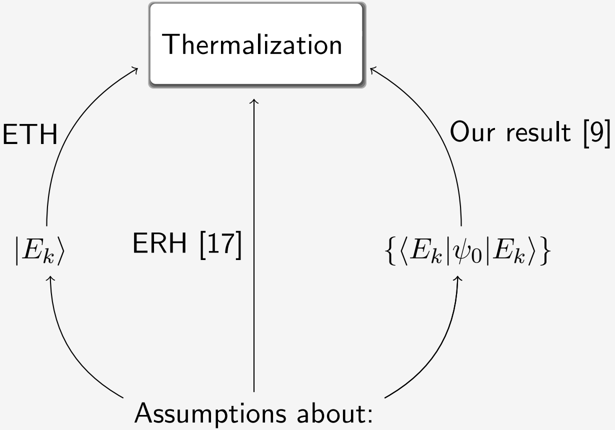

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Structure of the argument

Classical level counting à la Goldstein [18] with no interaction
$\mathscr{H}_0 = \mathscr{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_B$
→ Typicality arguments → Kinematic

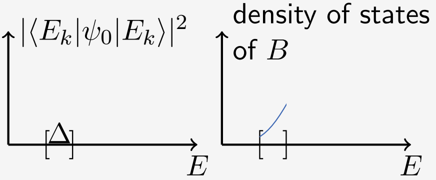
+

Perturbation theory for realistic weak coupling [9]
$\| \mathscr{H}_{SB} \|_\infty \ll k_B T$
→ Equilibration results → Dynamic

[9] A. Riera, C. Gogolin, and J. Eisert, PRL 108, 080402 (2012).
[18] S. Goldstein, J. Lebowitz, R. Tumulka, and N. Zanghì, PRL 96, 050403 (2006).

The result

$$\| \mathscr{H}_{SB} \|_\infty \gg \text{gaps}(\mathscr{H}_0)$$
$$\| \mathscr{H}_{SB} \|_\infty \ll k_B T \ll \Delta$$



$\Longrightarrow$ "Theorem" 2 (Theorem 2 in [9])

(Kinematic) Almost all pure states from a microcanonical subspace $[E, E + \Delta]$ are locally close to a Gibbs state.

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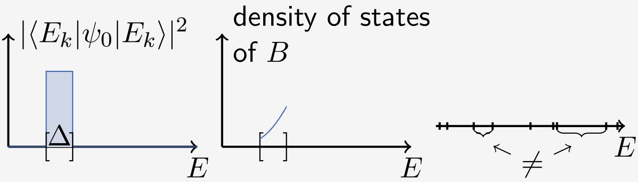

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(Dynamic) All initial states $\psi_{\sqcap,0}$ locally equilibrate towards a Gibbs state, even if they are initially far from equilibrium.

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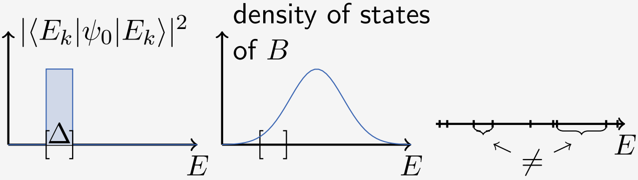

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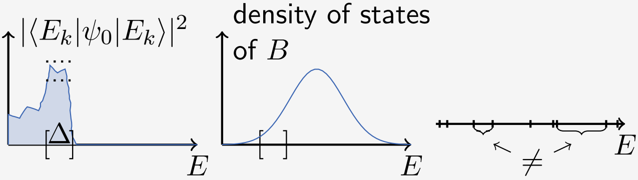

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Collaborators

Arnau Riera

Jens Eisert

Markus P. Müller

References

Thank you for your attention!

⟶ slides: www.cgogolin.de

[1] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne
Physical Review Letters **100** (2008) 30602.

[2] P. Reimann
Physical Review Letters **101** (2008) 190403.

[3] N. Linden, S. Popescu, A. Short, and A. Winter
Physical Review E **79** (2009) 61103.

[4] A. J. Short and T. C. Farrelly
New Journal of Physics **14** (2012) 013063.

[5] P. Reimann and M. Kastner
New Journal of Physics **14** (2012) 043020.

[6] C. Gogolin, M. P. Müller, and J. Eisert
Physical Review Letters **106** (2011) 40401.

[7] J. Gemmer, M. Michel, and G. Mahler,
Quantum Thermodynamics, vol. 784 of *Lecture Notes in Physics*, Springer Berlin Heidelberg 2009.

[8] A. Hutter and S. Wehner
Physical Review A **87** (2013) 012121.

[9] A. Riera, C. Gogolin, and J. Eisert
Physical Review Letters **108** (2012) 080402.

[10] C. Gogolin
Physical Review E **81** (2010) 051127.

[11] C. Kollath, A. Läuchli, and E. Altman
Physical Review Letters **98** (2007) no. 18, 180601.

[12] S. Manmana, S. Wessel, R. Noack, and A. Muramatsu
Physical Review Letters **98** (2007) 210405.

[13] M. Rigol, V. Dunjko, and M. Olshanii
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[14] M. C. Bañuls, J. I. Cirac, and M. B. Hastings
Physical Review Letters **106** (2011) 050405.

[15] M. Rigol
Physical Review Letters **103** (2009) 100403.

[16] J.-S. Caux and J. Mossel
Journal of Statistical Mechanics: Theory and Experiment **2011** (2011) P02023.

[17] T. N. Ikeda, Y. Watanabe, and M. Ueda
Physical Review E **84** (2011) 021130.

[18] S. Goldstein, J. Lebowitz, R. Tumulka, and N. Zanghì
Physical Review Letters **96** (2006) 50403.