Measure concentration in Hilbert space

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Quantum mechanical typicality and the foundations of Thermodynamics

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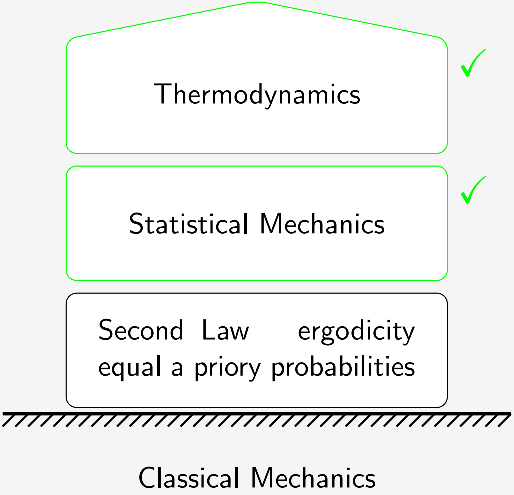
2009-12-18

New foundation for Statistical Mechanics



[2, 3]

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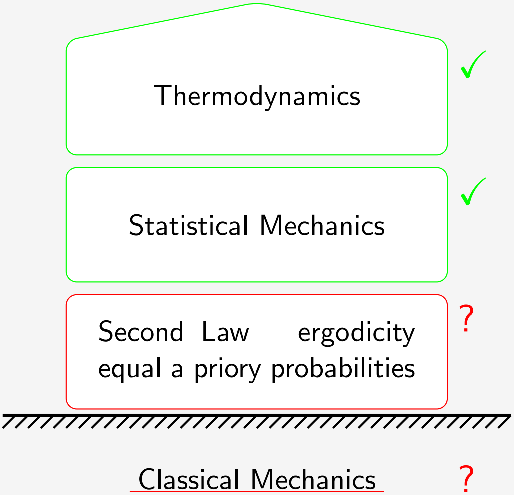

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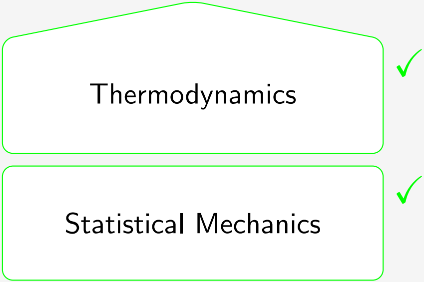

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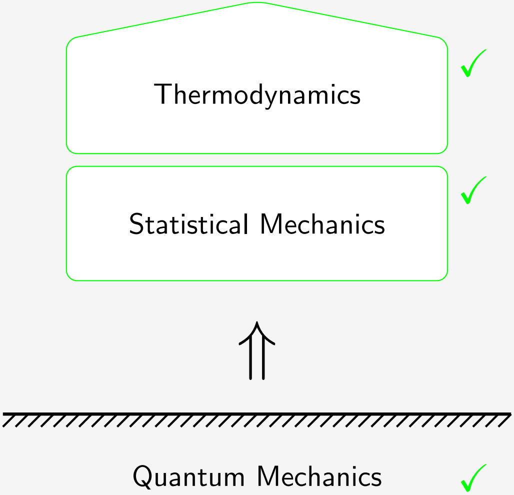

[2, 3]

What is the equal a priory probability postulate?

The fundamental postulate in Statistical Mechanics:

Equal a priory probability Postulate

We want to calculate the expectation value $\langle A \rangle$ but have only limited knowledge about the system.

Then:

$$\langle A \rangle = \langle A \rangle_{\mathrm{mc}} := \text{average over all compatible states with equal weights}$$

What is the Second Law?

- Clausius

  Heat generally can not spontaneously flow from a material at lower temperature to a material at higher temperature.

[1,en.wikipedia.org]

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  It is impossible to convert heat completely into work in a cyclic process.

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  It is impossible to convert heat completely into work in a cyclic process.

- Boltzmann (H-Theorem)

  The entropy in a closed system can not decrease. It stays constant only for reversible processes.

[1,en.wikipedia.org]

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Quantum Mechanics on one slide

Quantum Mechanics on one slide

- Pure Quantum Mechanics

$$|\psi\rangle \in \mathcal{H} \qquad A = A^\dagger$$

$$\langle\psi|\psi\rangle = 1 \qquad \langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$

$$|\psi_t\rangle = U_t\,|\psi_0\rangle \qquad U_t = \mathrm{e}^{-\mathrm{i}\,\mathscr{H}\,t}$$

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$$\rho, \psi \in \mathcal{M}(\mathcal{H}) \qquad \psi = |\psi\rangle\langle\psi|$$

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mixtures: $\rho = p\,\psi_1 + (1-p)\,\psi_2$

Measure concentration in Hilbert space

Choosing random states

- Mathematical construction

Haar measure on $SU(n)$ $\longrightarrow$ "uniform" distribution

$$\mu(V) = \mu(U\,V) \qquad U\,|0\rangle = |\psi\rangle \qquad \mu\{|\psi\rangle\} = \mu\{U\,|\psi\rangle\}$$

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- Explicit construction
  1. expand in basis: $|\psi\rangle = \sum_i (a_i + \mathrm{i}\ b_i)\,|i\rangle \qquad \langle i|j\rangle = \delta_{ij}$
  2. choose a_i and b_i from a normal distribution
  3. normalize $1 = \sum_i |a_i|^2 + |b_i|^2 \qquad \langle\psi|\psi\rangle = 1$

Quantum states as vectors in $\mathbb{R}^{2d}$

- States $|\psi\rangle$ can be thought of as vectors in $\mathbb{R}^{2d}$:

$$|\psi\rangle = \sum_i (a_i + \mathrm{i}\ b_i)\,|i\rangle \in \mathcal{H} \quad \longleftrightarrow \quad \vec{x} \in \mathbb{R}^{2d}$$

$$x_{2i}(\psi) = a_i \qquad\qquad x_{2i+1}(\psi) = b_i$$

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- Norm, metric and the uniform measure are preserved:

$$\|\vec{x}\| = \||\psi\rangle\|_2 = \sqrt{\langle\psi|\psi\rangle}$$
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$\Longrightarrow$ Normalized states lie on the surface S^{2d-1} of a hypersphere in $\mathbb{R}^{2d}$.

Levy's lemma

Lemma 1 (Levy's lemma)

Let $f : S^{d-1} \to \mathbb{R}$ *have a finite Lipschitz constant*

$$\eta = \sup_{\vec{x}_1, \vec{x}_2} \frac{|f(\vec{x}_1) - f(\vec{x}_2)|}{\|\vec{x}_1 - \vec{x}_2\|},$$

with respect to the euclidean norm $\|\cdot\|$.
Then, for uniformly random points $\vec{x} \in S^{d-1}$,

$$\mu\{|f(\vec{x}) - \langle f \rangle| \geq \epsilon\} \leq 2\, \mathrm{e}^{-\frac{C\, d\, \epsilon^2}{\eta^2}},$$

where $C = (9\,\pi^3)^{-1}$.

[5]

Typicality of expectation values

Theorem 2

Let $\mathcal{H}_R \subseteq \mathcal{H},\ d_R = \dim(\mathcal{H}_R)$ *and* Π_R *the projector onto* $\mathcal{H}_R$*.*
For randomly chosen pure states $\psi = |\psi\rangle\langle\psi|$

$$\mu\left\{\left|\operatorname{Tr}[A\,\psi] - \operatorname{Tr}[A\frac{\Pi_R}{d_R}]\right| \geq \epsilon\right\} \leq 2\,\mathrm{e}^{-\frac{C\,d_R\,\epsilon^2}{\|A\|_\infty^2}},$$

for every $\epsilon > 0$*, where* $C = (36\,\pi^3)^{-1}$.

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for every $\epsilon > 0$*, where* $C = (36\,\pi^3)^{-1}$*.*

$\Longrightarrow$ Theorem 2 justifies the equal a priory probability postulate!

Typicality of expectation values

Proof.

- Define:

$$f_A(\psi) = \mathrm{Tr}[A\,\psi] \quad f : S^{2d-1} \to \mathbb{R}$$

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- Define:

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- Calculate the expectation value:

$$\langle f_A(\psi) \rangle_\psi = \langle \mathrm{Tr}[A\,\psi] \rangle_\psi = \mathrm{Tr}[A\,\langle \psi \rangle_\psi] = \mathrm{Tr}[A \frac{\Pi_R}{d_R}]$$

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Proof.

- Define:

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- Calculate the expectation value:

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- Bound the Lipschitz constant:

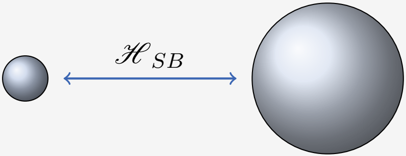
$$\begin{aligned}|f_A(\psi_1) - f_A(\psi_2)| &= |\,\mathrm{Tr}[A(\psi_1 - \psi_2)]| \\ &\leq \|A\|_\infty \, \||\psi_1\rangle + |\psi_2\rangle\|_2 \, \||\psi_1\rangle - |\psi_2\rangle\|_2 \\ &\leq 2\,\|A\|_\infty \, \||\psi_1\rangle - |\psi_2\rangle\|_2\end{aligned}$$

□

Subsystem equilibration

Setup

System, $\mathcal{H}_S, \mathscr{H}_S$ Bath, $\mathcal{H}_B, \mathscr{H}_B$



$$\rho_t^S = \mathrm{Tr}_B[\psi_t] \qquad \rho_t^B = \mathrm{Tr}_S[\psi_t]$$

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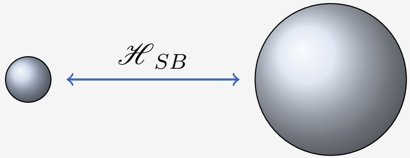

$$\rho_t^S = \mathrm{Tr}_B[\psi_t] \qquad \rho_t^B = \mathrm{Tr}_S[\psi_t]$$

$$\mathrm{Tr}[(A_S \otimes \mathbb{1}_B)\psi_t] = \mathrm{Tr}[A_S\, \rho_t^S]$$

reduced state $\rightarrow$ locally observable

A very weak assumption on the Hamiltonian

Definition

A Hamiltonian has non-degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\Longrightarrow k = l \wedge m = n \text{ or } k = m \wedge l = n$$

Two definitions

- Trace distance

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- Time average

$$\begin{aligned}\overline{\rho_t} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \rho_t\,dt \\ \overline{f(\rho_t)} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\rho_t)\,dt\end{aligned}$$

Equilibration

Theorem 3

Let $\mathscr{H}$ have non-degenerate energy gaps.
Then for every $\psi_0 = |\psi_0\rangle\langle\psi_0|$

$$\overline{\mathcal{D}(\rho_t^S, \overline{\rho_t^S})} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}.$$

$d_S = \dim(\mathcal{H}_S)$ $\qquad$ $d^{\text{eff}} \sim$ # energy eigenstates in $|\psi_0\rangle$

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Equilibration

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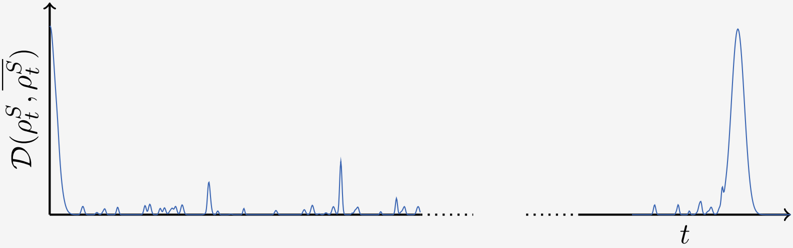
$d_S = \dim(\mathcal{H}_S)$ $\quad$ $d^{\text{eff}} \sim$ # energy eigenstates in $|\psi_0\rangle$

$\Longrightarrow$ If $d^{\text{eff}} \gg d_S^2$ then ρ_t^S equilibrates.

[6]

Statistical equilibration

$$\overline{\mathcal{D}(\rho_t^S, \overline{\rho_t^S})} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}$$



$\Longrightarrow$ No conflict with time reversal invariance!

Typical effective dimension

Theorem 4

For random $\psi_0 = |\psi_0\rangle\langle\psi_0| \in \mathcal{H}_R$ *with* $d_R = \dim(\mathcal{H}_R)$

$$\mu \left\{ d^{\text{eff}} < \frac{d_R}{4} \right\} \leq 2\, \mathrm{e}^{-c\sqrt{d_R}}$$

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$$\mu\left\{d^{\text{eff}} < \frac{d_R}{4}\right\} \leq 2\,\mathrm{e}^{-c\sqrt{d_R}}$$

$\Longrightarrow$ If d_R is large then d^{eff} is large.

[6]

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Typical states of large quantum systems

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- and their subsystems equilibrate.

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Thank you for your attention!

$\longrightarrow$ beamer slides: http://www.cgogolin.de