Simulability of open quantum system dynamics

"A dissipative quantum Church-Turing theorem"

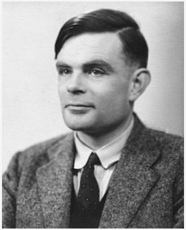
M. Kliesch, T. Barthel, C. Gogolin, M. Kastoryano, and J. Eisert

Dahlem Center for Complex Quantum Systems, Freie Universität Berlin

QCCC Workshop 2011

Simulability

What is (efficiently) simulatable?



[1] A. M. Turing, Proc. London Math. Soc. 42 (1937) no. 230, 230–265

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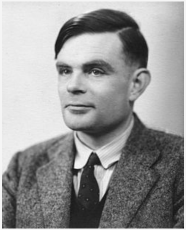

Here: Quantum many body dynamics

- On a quantum computer?
- On a classical computer?

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Preliminaries

Unitary vs. Liouvillian dynamics

Unitary:

equation of motion: $\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}[\mathrm{H}, \rho(t)]$

Unitary vs. Liouvillian dynamics

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Propagator for $t \geq s \geq 0$

$$\rho(t) = T_{\mathcal{L}}(t, s)(\rho(s))$$

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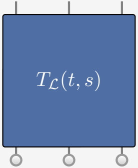

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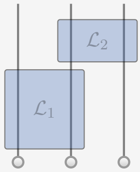

Distinguishability of propagators

Distinguishability of density matrices:

$$\|\rho - \sigma\|_1 = \max_{0 \leq A \leq \mathbb{1}} \operatorname{tr}(A(\rho - \sigma))$$

Worst case estimate for propagators:

$$\left\|T - T'\right\|_{1\to 1} := \sup_{\|\rho\|_1 = 1} \|T(\rho) - T'(\rho)\|_1$$

Trotterization of Liouvillian dynamics

k-local Liouvillian dynamics

$$\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda(\rho) = -\mathrm{i}[\mathrm{H}_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu}\rho L_{\Lambda,\mu}{}^\dagger - \{L_{\Lambda,\mu}{}^\dagger L_{\Lambda,\mu}, \rho\}$$

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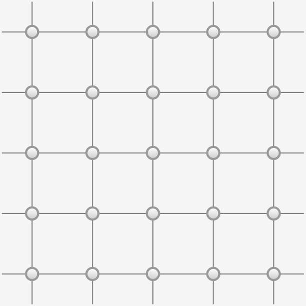
Assumptions:

- N sites with finite local dimension d



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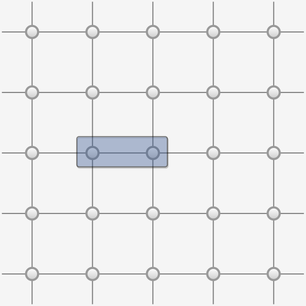

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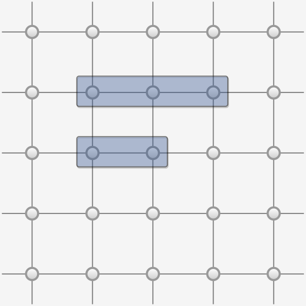

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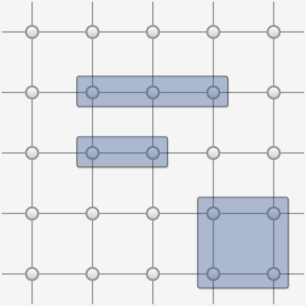

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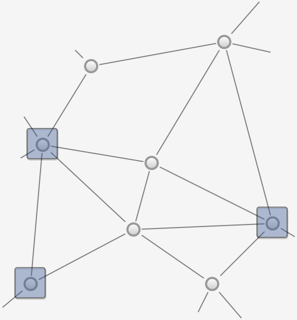

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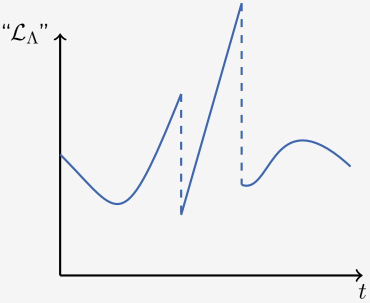

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- N sites with finite local dimension d
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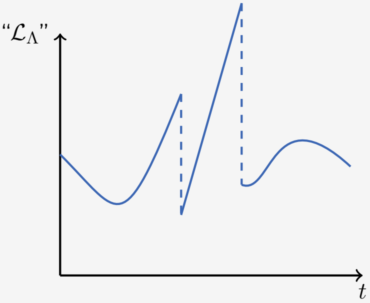
- arbitrary time dependence
- $\|\mathrm{H}_\Lambda\|_\infty$ and $\|L_{\Lambda,\mu}\|_\infty$ bounded independent of N



Trotterization – what it is and how we get there

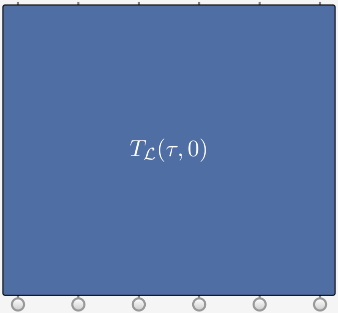
What we are aiming for:

$$T_{\mathcal{L}}(\tau,0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{\mathcal{L}_\Lambda}(\Delta t\, j, \Delta t(j-1))$$

What we have to do:

1. Decompose $T_{\mathcal{L}}(\tau,0)$ in time slices.
2. Approximate each time slice by applying local Liouvillians sequentially.

Trotterization of k-local Liouvillian dynamics



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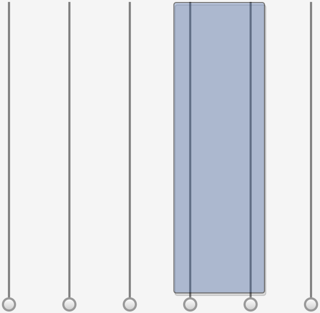

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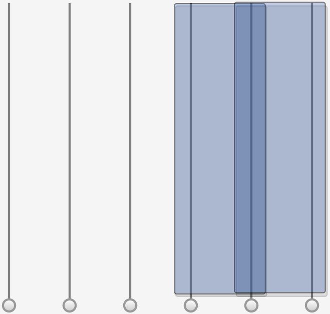

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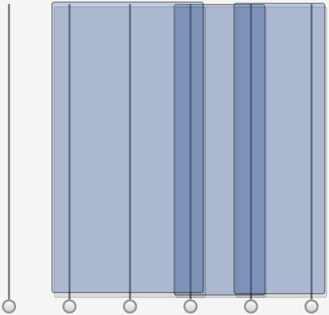

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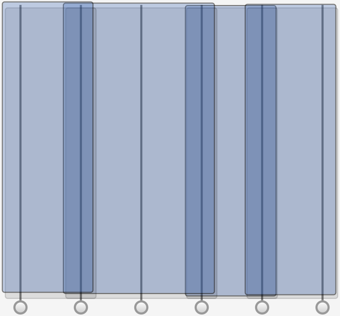

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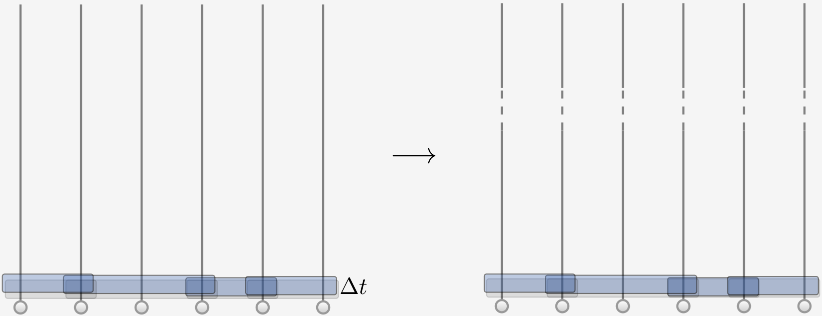

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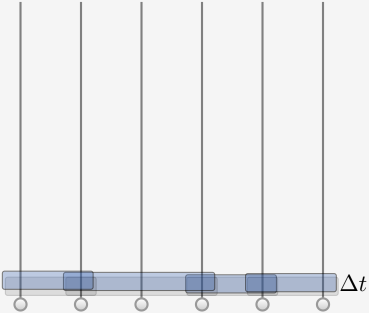

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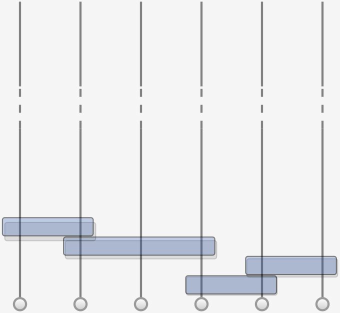

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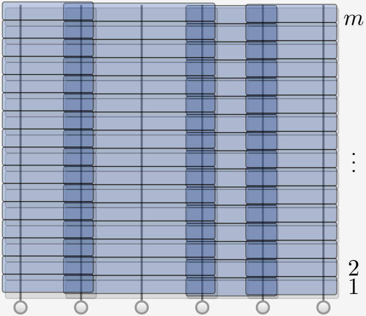

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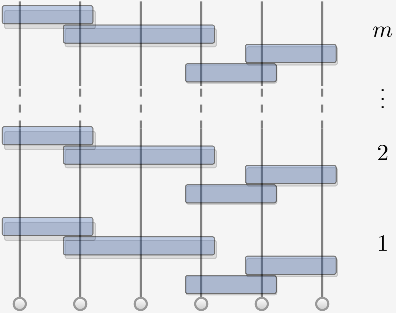

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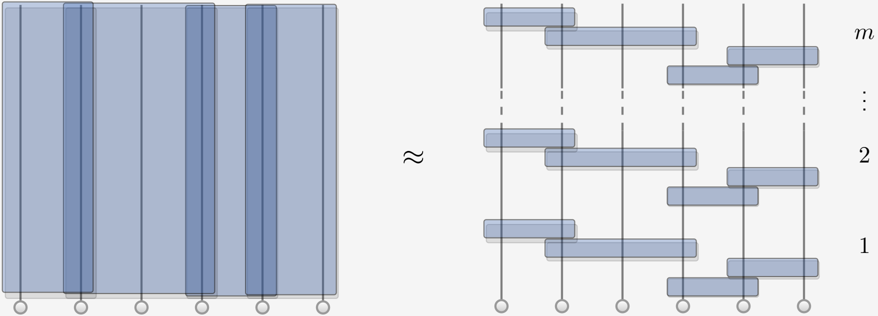

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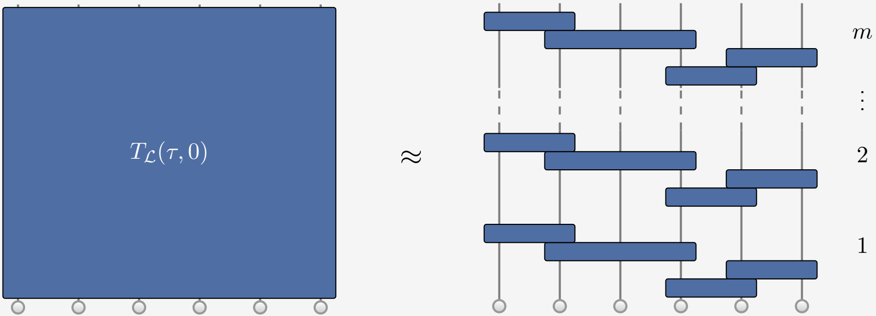

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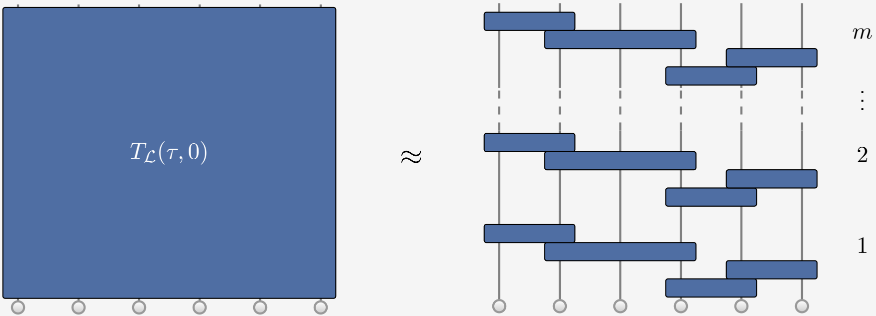

$$m = \mathcal{O}\left(\frac{d^{2k}\, K^2\, \tau^2}{\epsilon}\right)$$

A Trotter Formula for Liouvillian dynamics

Theorem 1 (Trotter decomposition [2])

Let $\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda$ be a piecewise continuous time dependent, k-local Liouvillian acting on N subsystems of dimension d, then

$$\left\| T_{\mathcal{L}}(\tau, 0) - \prod_{j=1}^{m} \prod_{\Lambda} T_{\mathcal{L}_\Lambda}\left(\Delta t\, j, \Delta t\,(j-1)\right) \right\|_{1\to 1} \leq c\, K^2\, \tau\, \Delta t\; \mathrm{e}^{b\, \Delta t},$$

with $b \in \mathrm{O}(d^k)$, $c \in \mathrm{O}(d^{2k})$.

[2] M. Kliesch, et al., PRL 107 (2011) 120501

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$$\left\| T_{\mathcal{L}}(\tau, 0) - \prod_{j=1}^{m} \prod_{\Lambda} T_{\mathcal{L}_\Lambda^{\mathrm{av}}}(\Delta t\, j, \Delta t\,(j-1)) \right\|_{1\to 1} \leq c\, K^2\, \tau\, \Delta t\ \mathrm{e}^{b\, \Delta t},$$

with $b \in \mathrm{O}(d^k)$, $c \in \mathrm{O}(d^{2k})$.

$$T_{\mathcal{L}_\Lambda^{\mathrm{av}}}(\Delta t\, j, \Delta t\,(j-1)) = \exp(\Delta t\, \mathcal{L}_\Lambda^{\mathrm{av}}) \qquad \mathcal{L}_\Lambda^{\mathrm{av}} = \Delta t \int_{\Delta t\,(j-1)}^{\Delta t\, j} \mathcal{L}_\Lambda \mathrm{d}t$$

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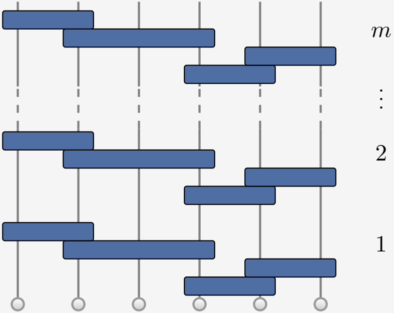
Stinespring and Solovay-Kitaev



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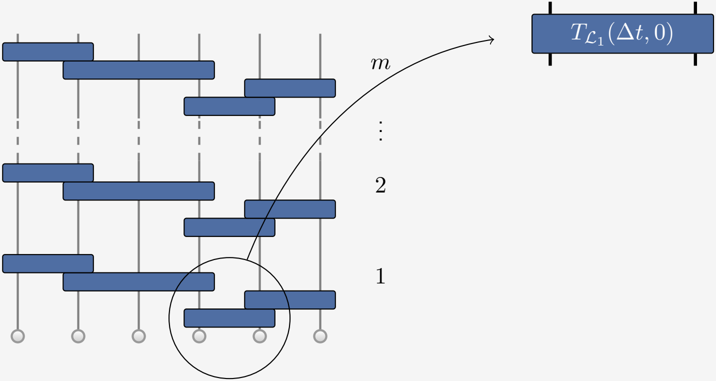

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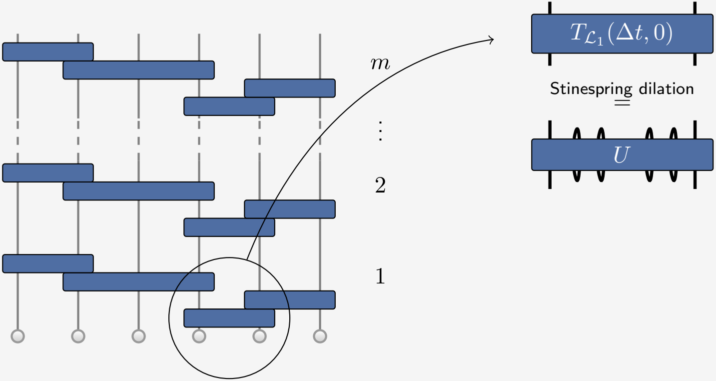

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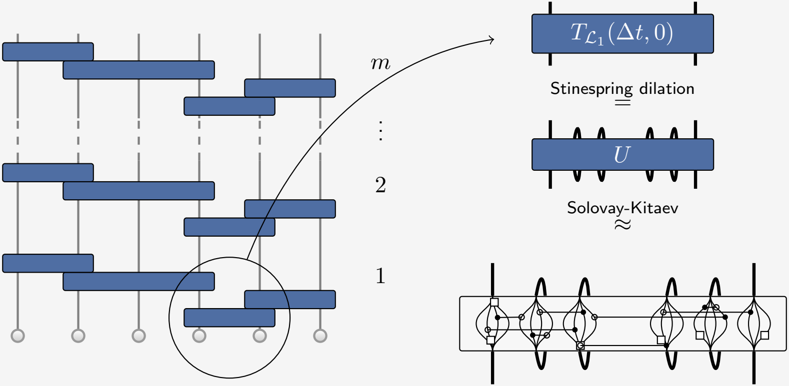
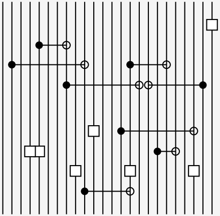


Implications

Implication 1

Power of dissipative quantum computing [3, 2]

Dissipative quantum computing with k-local, arbitrary time dependent Liouvillian dynamics is exactly as powerful as the circuit model.



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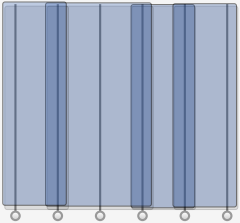

[3] F. Verstraete, M. Wolf, and I. Cirac, Nature Physics, Vol 5, (2009) 633
[2] M. Kliesch, et al., PRL 107 (2011) 120501

Implication 2

Limits on efficient state preparation [2]

Even with arbitrary time-dependent k-local Liouvillian dynamics one can only reach exponentially few states after polynomial time.

[4] D. Poulin, A. Qarry, R. Somma, and F. Verstraete, PRL 106 (2011) 170501
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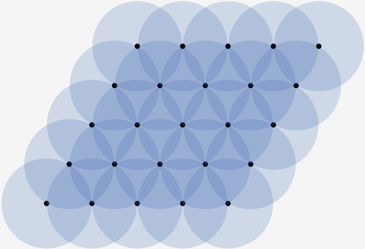

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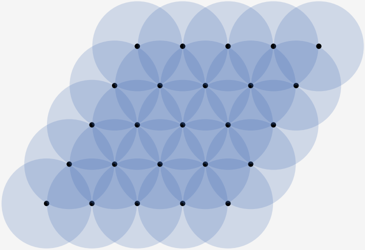

Smallest ϵ-nets:

$$\Omega\left(\exp(d^N)\right)$$

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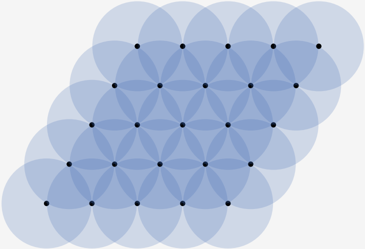

Smallest ϵ-nets:

$$\Omega\left(\exp(d^N)\right)$$

Number of circuits for ϵ-approximation:

$$\mathrm{O}\left(\exp(N^{3k+2}\tau^4)\right)$$

[4] D. Poulin, A. Qarry, R. Somma, and F. Verstraete, PRL 106 (2011) 170501
[2] M. Kliesch, et al., PRL 107 (2011) 120501

Implication 3

Simulation on classical computers [2]

For fixed τ dissipative dynamics can be simulated efficiently in N on classical computers.

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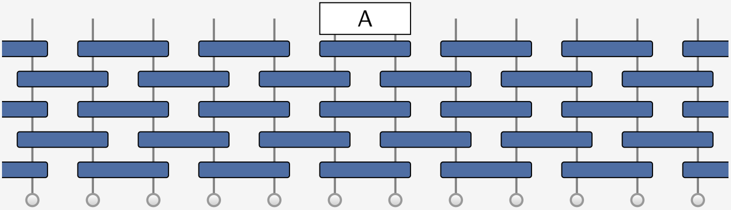

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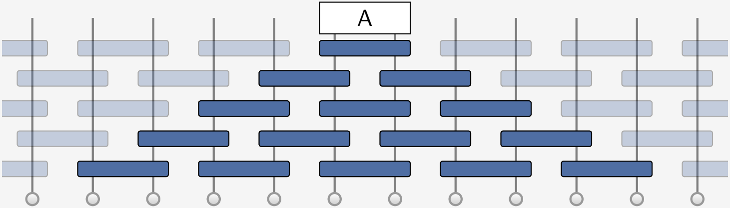

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Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

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- k-local Liouvillian dynamics
- arbitrary time dependence

[2] M. Kliesch, et al., PRL 107 (2011) 120501

Implication 4

Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Remember the assumption we made:

- N sites with finite local dimension d
- k-local Liouvillian dynamics
- arbitrary time dependence

Arguably the most broadest setting that allows efficient simulation.

[2] M. Kliesch, et al., PRL 107 (2011) 120501

Summary

$$T_{\mathcal{L}}(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{\mathcal{L}_\Lambda}(\Delta t\, j, \Delta t\,(j-1))$$

- k-local Liouvillian dynamics can be trotterized

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$$T_{\mathcal{L}}(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{\mathcal{L}_\Lambda}(\Delta t\, j, \Delta t\, (j-1))$$

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- Dissipative quantum computing is no more powerful than the circuit model
- Most states can not be prepared efficiently
- k-local Liouvillian dynamics can be simulated classically (efficient in N, inefficient in τ)
- A strong quantum Church-Turing theorem holds

Collaborators

Freie Universität Berlin


Martin Kliesch


Thomas Barthel


Jens Eisert

Technische Universität München


Michael Kastoryano

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Thank you for your attention!

⟶ slides: www.cgogolin.de

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