

Under what conditions do quantum systems thermalize?

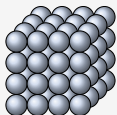
New insights from quantum information theory

Christian Gogolin, Markus Müller, Arnau Riera, and Jens Eisert

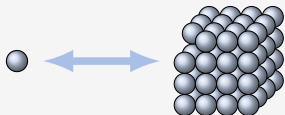
University of Potsdam

QIP 2011 - Singapore

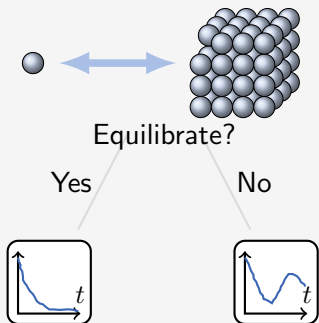
How QIT helps understand many particle dynamics



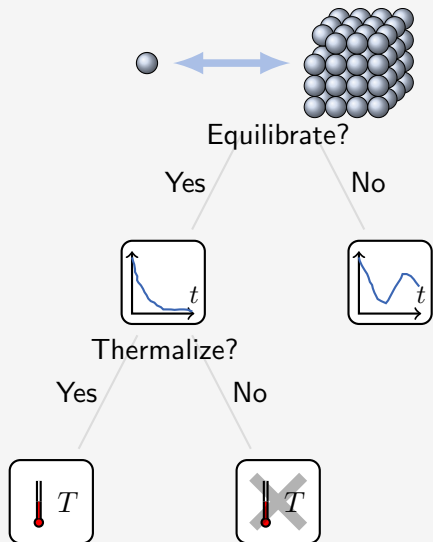
How QIT helps understand many particle dynamics



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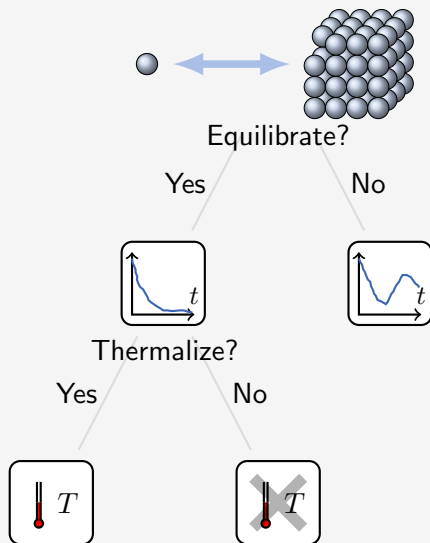


How QIT helps understand many particle dynamics



- Information propagation/
Lieb-Robinson bounds
- Haar-measure averages/
Concentration of measure

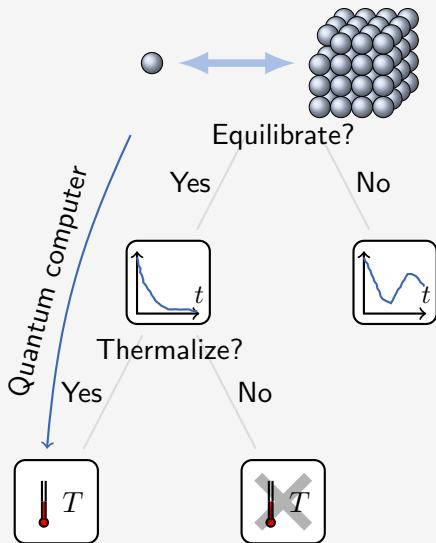
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Setup

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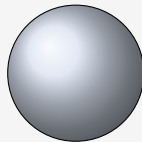
System, $\mathcal{H}_S, \mathcal{H}_S$

$$d_S = \dim(\mathcal{H}_S)$$



Bath, $\mathcal{H}_B, \mathcal{H}_B$

$$d_B \gg d_S$$



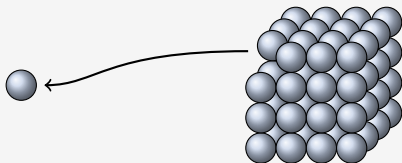
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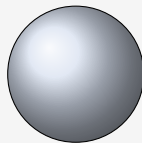
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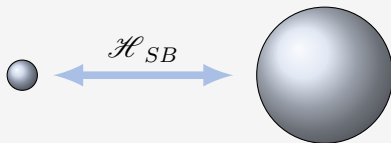
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$$\mathcal{H} = \mathcal{H}_S \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{H}_B + \mathcal{H}_{SB}$$

$$\frac{d\psi_t}{dt} = i[\psi_t, \mathcal{H}]$$

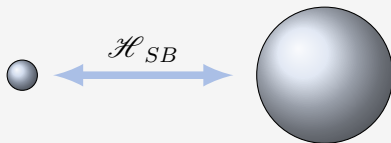
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$$\psi_t^S = \text{Tr}_B[\psi_t]$$

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$$\frac{d\psi_t}{dt} = i[\psi_t, \mathcal{H}]$$

Equilibration

Equilibration

Theorem 1 (Equilibration [1])

If \mathcal{H} has *non-degenerate energy gaps*, then for every $\psi_0 = |\psi_0\rangle\langle\psi_0|$ there exists a ω^S such that:

$$\overline{\mathcal{D}(\psi_t^S, \omega^S)} \leq \frac{1}{2} \sqrt{\frac{d_S^2}{d^{\text{eff}}}}.$$

Equilibration

Theorem

If \mathcal{H} has
there exists

Non-degenerate energy gaps

\mathcal{H} has non-degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\implies k = l \wedge m = n \quad \vee \quad k = m \wedge l = n$$

Intuition: Sufficient for \mathcal{H} to be fully interactive

$$\mathcal{H} \neq \mathcal{H}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{H}_2$$

$\langle \psi_0 |$

Equilibration

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[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

Equilibration

Theorem 1 (Equilibration [1])

If \mathcal{H} has
there exists

Effective dimension

$$d^{\text{eff}} = \frac{1}{\sum_k |\langle E_k | \psi_0 \rangle|^4}.$$

Intuition: Dimension of supporting energy subspace

$\langle \psi_0 |$

Equilibration

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If \mathcal{H} has *non-degenerate energy gaps*, then for every $\psi_0 = |\psi_0\rangle\langle\psi_0|$ there exists a ω^S such that:

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\implies If $d^{\text{eff}} \gg d_S^2$ then ψ_t^S *equilibrates*.

Maximum entropy principle

Theorem 2 (Maximum entropy principle [2])

If $\text{Tr}[A \psi_t]$ equilibrates, it equilibrates towards its time average

$$\overline{\text{Tr}[A \psi_t]} = \text{Tr}[A \overline{\psi_t}] = \text{Tr}[A \omega],$$

and

$$\omega = \sum_k \pi_k \psi_0 \pi_k$$

is the *dephased* state that maximizes the von Neumann entropy, given all conserved quantities.

[2] C. Gogolin, M. P. Mueller, and J. Eisert, (to appear in PRL), 1009.2493

Maximum Entropy Principle

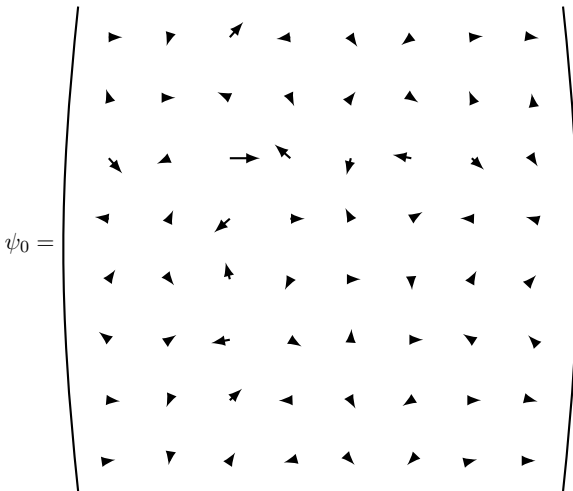
Theorem

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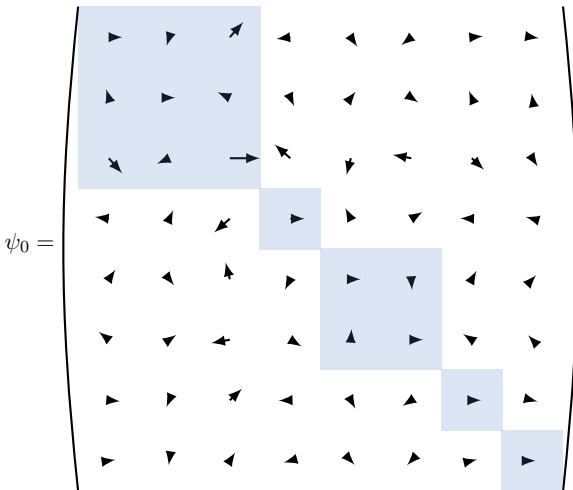
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is the dep
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Time averaging



Time averaging



Theorem

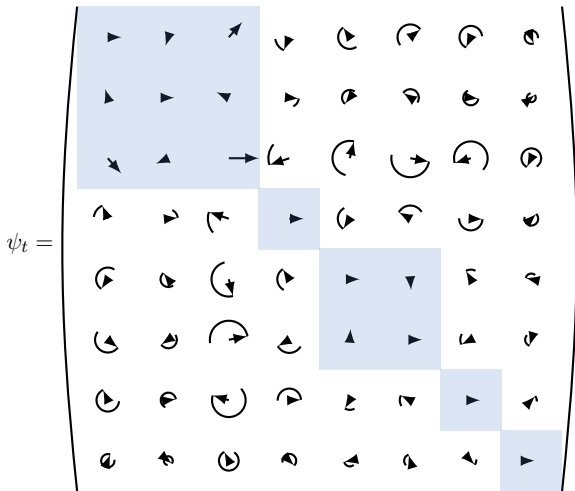
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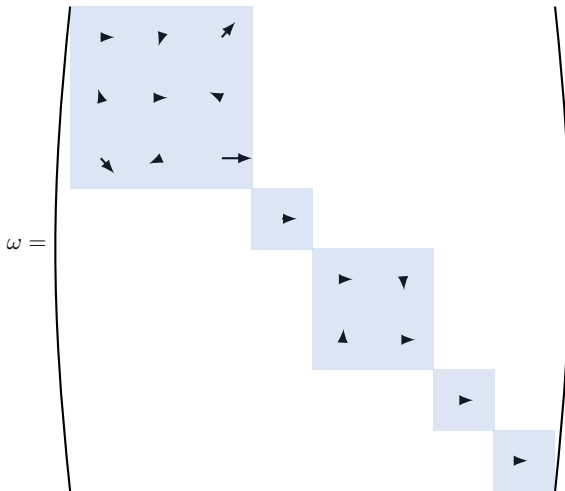
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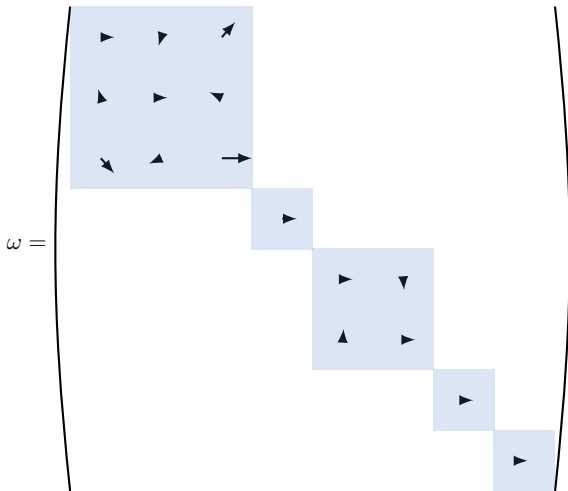
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$\psi_0 \mapsto \omega$ is a **pinching** $\Rightarrow \omega$ **maximizes entropy**.

Theorem

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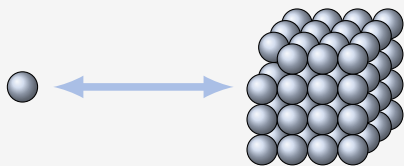
$$\omega = \sum_k \pi_k \psi_0 \pi_k$$

is the *dephased state* that *maximizes the von Neumann entropy*, given all conserved quantities.

⇒ Maximum entropy principle from pure quantum dynamics.

Thermalization

Thermalization is a complicated process



Thermalization implies:

- 1 Equilibration [1]
- 2 Subsystem initial state independence [2]
- 3 Weak bath state dependence [4]
- 4 Diagonal form of the subsystem equilibrium state [7]
- 5 Gibbs state $\omega^S = \text{Tr}_B[\omega] \approx e^{-\beta \mathcal{H}^S}$ [4]

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

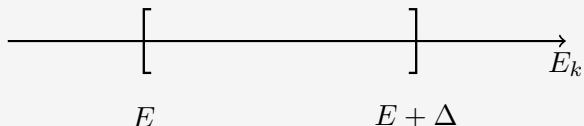
[2] C. Gogolin, M. P. Mueller, and J. Eisert, (to appear in PRL), 1009.2493

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

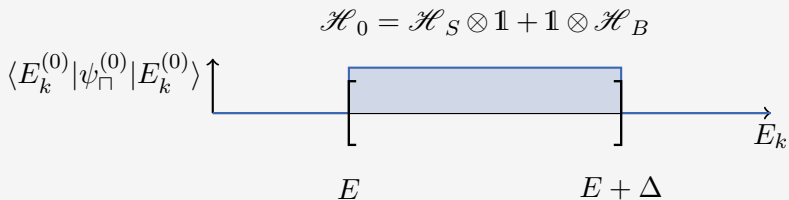
[7] C. Gogolin, PRE 81 (2010) no. 5, 051127

Level counting with no coupling

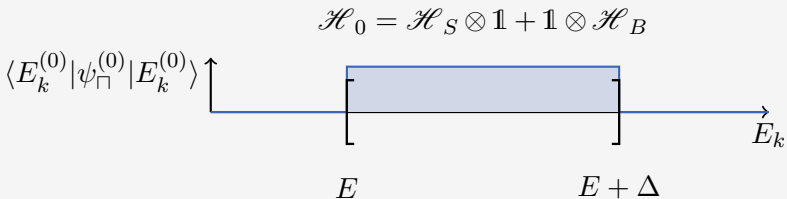
$$\mathcal{H}_0 = \mathcal{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_B$$



Level counting with no coupling



Level counting with no coupling

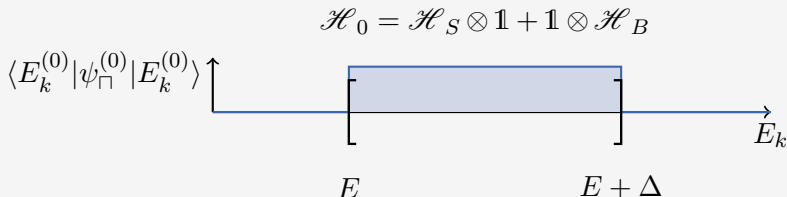


Well known fact [3]:

$$\omega_{\square}^{S(0)} \stackrel{\text{no coupling}}{\propto} \sum_k \Omega_{\Delta}^B(E - E_k^S) |E_k^S\rangle \langle E_k^S|$$

$\#$ bath states in $[E - E_k^S, E - E_k^S + \Delta]$

Level counting with no coupling



Well known fact [3]:

$$\omega_{\square}^{S(0)} \stackrel{\text{no coupling}}{\propto} \sum_k \Omega_{\Delta}^B(E - E_k^S) |E_k^S\rangle \langle E_k^S| \approx \sum_k e^{-\beta E_k^S} |E_k^S\rangle \langle E_k^S|$$

\nwarrow # bath states in $[E - E_k^S, E - E_k^S + \Delta]$
 \uparrow exponentially dense spectrum

Perturbative coupling $\| \mathcal{H}_{SB} \|_{\infty} < \text{gaps}(\mathcal{H}_0) \dots$

- \dots is unrealistic as the spectrum of \mathcal{H}_0 becomes exponentially dense.

Perturbative coupling $\| \mathcal{H}_{SB} \|_{\infty} < \text{gaps}(\mathcal{H}_0) \dots$

- ... is **unrealistic** as the spectrum of \mathcal{H}_0 becomes **exponentially dense**.
- ... provably prevents thermalization because

perturbative coupling

\Downarrow [2]

effective entanglement in the eigenbasis $R(\psi_0)$ is small

\Downarrow [2]

absence of initial state independence.

$$\mathcal{D}(\omega^{S(1)}, \omega^{S(2)}) \geq \mathcal{D}(\psi_0^{S(1)}, \psi_0^{S(2)}) - R(\psi_0^{S(1)}) - R(\psi_0^{S(2)})$$

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\Rightarrow Refutes wide spread believe that “**non-integrable models thermalize.**”

Realistic weak coupling

- Naive perturbation theory fails.
- Realistic weak coupling: $\text{gaps}(\mathcal{H}_0) \ll \|\mathcal{H}_{SB}\|_\infty \ll \Delta$

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

Realistic weak coupling

- Naive perturbation theory fails.
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Theorem 3 (Corollary of a theorem from [4])

If $\|\mathcal{H}_{SB}\|_\infty \ll \Delta$ the dephased states $\omega_\square^{S(0)}$ and ω_\square^S are *close to each other* in the sense that

$$\mathcal{D}(\omega_\square^S, \omega_\square^{S(0)}) \lesssim 3\sqrt{\frac{\|\mathcal{H}_{SB}\|_\infty}{2\Delta}}.$$

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

Consequences

$$\implies \text{Tr}_B[\omega_\square] = \omega_\square^S \approx \rho_{\text{Gibbs}}^S$$

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Consequences

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“Theorem” 5 (Conclusion [4])

Assume $\Omega_\Delta^B(E - E_k^S)$ becomes **exponentially dense** to higher energies.

(Kinematic)

If the coupling is **weak** $\|\mathcal{H}_{SB}\|_\infty \ll \Delta$, **almost all pure states** from a microcanonical subspace $[E, E + \Delta]$ are **locally close** to a **Gibbs state**.

(Dynamic)

If the Hamiltonian has **non-degenerate energy gaps** all initial states $\psi_{\square,0}$ with a **flat energy distribution** in $[E, E + \Delta]$ locally equilibrate towards a **Gibbs state**, even if they are **initially far from equilibrium**.

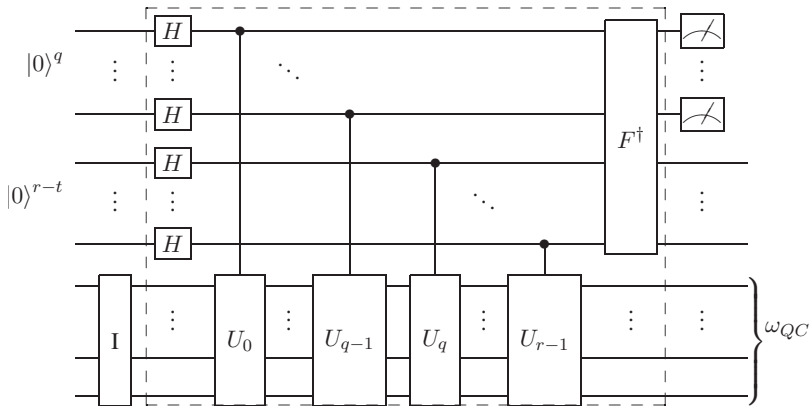
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A quantum algorithm for Gibbs state preparation

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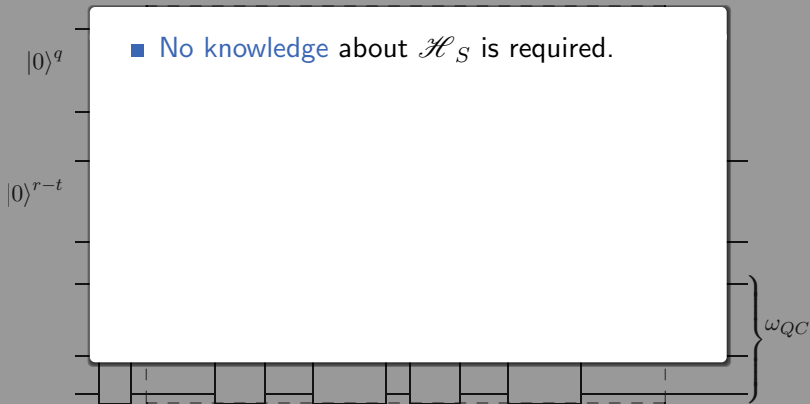
A quantum algorithm for Gibbs state preparation

Quantum circuit



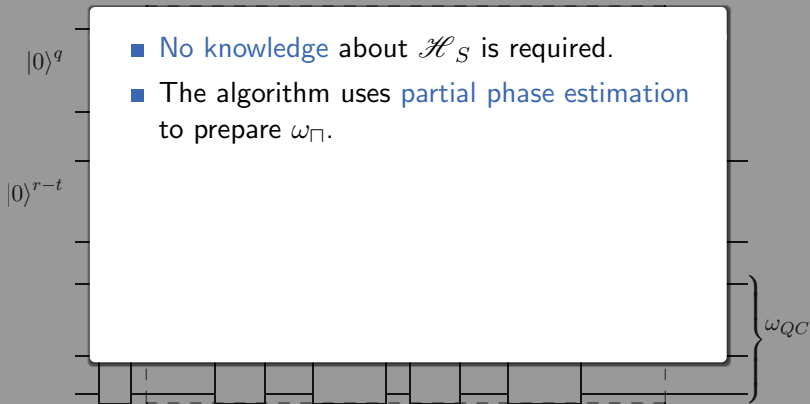
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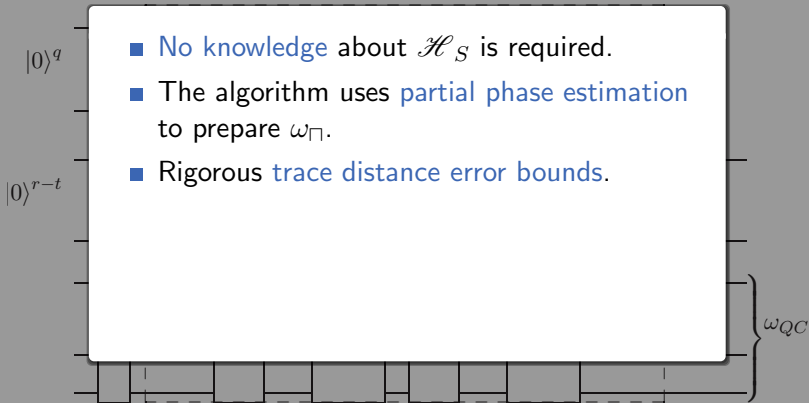
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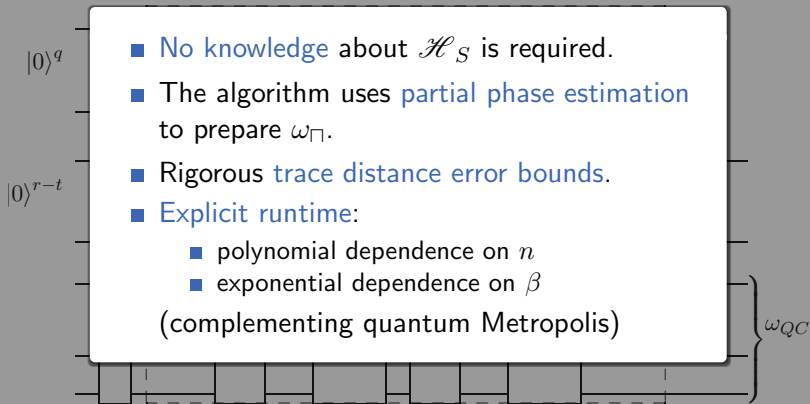
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Outlook

And there is more . . .

What I didn't talk about:

- Thermalization in [exactly solvable models](#) [5, 6]
- A strong connection to [decoherence](#) [7]
- [Measure concentration](#) [8, 1, 9, 10]

The major open question:

- [Time scales](#). How long does it take to equilibrate/thermalize/decohere?

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

[5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, PRL 100 (2008) 030602

[6] M. Cramer and J. Eisert, NJP 12 (2010) 055020

[7] C. Gogolin, PRE 81 (2010) no. 5, 051127

[8] S. Popescu, A. J. Short, and A. Winter, Nature Physics 2 (2006) no. 11, 754

[9] M. Mueller, D. Gross, and J. Eisert, 1003.4982

[10] C. Gogolin, Master's thesis, 2010, 1003.5058

Collaborators



Markus P. Müller



Arnau Riera



Jens Eisert



Peter Janotta



Haye Hinrichsen



Andreas Winter

References

Thank you for your attention!

→ slides: www.cgogolin.de

- [1] N. Linden, S. Popescu, A. J. Short, and A. Winter, "Quantum mechanical evolution towards thermal equilibrium", *Physical Review E* 79 (2009) no. 6, 061103.
- [2] C. Gogolin, M. P. Mueller, and J. Eisert, "Absence of thermalization in non-integrable systems", (to appear in *Physical Review Letters*) (2010) , 1009.2493v1.
- [3] S. Goldstein, "Canonical Typicality", *Physical Review Letters* 96 (2006) no. 5, 050403.
- [4] A. Riera, C. Gogolin, and J. Eisert, "Gibbs states, exact thermalization of quantum systems, and a certifiable quantum algorithm for preparing thermal states", (*unpublished*) (2010) , 1101.????
- [5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, "Exact Relaxation in a Class of Nonequilibrium Quantum Lattice Systems", *Phys. Rev. Lett.* 100 (2008) 030602.
- [6] M. Cramer and J. Eisert, "A quantum central limit theorem for non-equilibrium systems: exact local relaxation of correlated states", *New J. Phys* 12 (2010) 055020.
- [7] C. Gogolin, "Environment-induced super selection without pointer states", *Physical Review E* 81 (2010) no. 5, 051127.
- [8] S. Popescu, A. J. Short, and A. Winter, "Entanglement and the foundations of statistical mechanics", *Nature Physics* 2 (2006) no. 11, 754.
- [9] M. Mueller, D. Gross, and J. Eisert, "Concentration of measure for quantum states with a fixed expectation value", 1003.4982v2.
- [10] C. Gogolin, "Pure State Quantum Statistical Mechanics", Master's thesis, Julius-Maximilians Universität Würzburg, 2010. <http://arxiv.org/abs/1003.5058>.