Absence of thermalization in non-integrable systems

Christian Gogolin, Arnau Riera, Markus Müller, and Jens Eisert

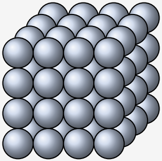
Dahlem Center for Complex Quantum Systems, Freie Universität Berlin

Workshop "Many-Body Quantum Dynamics in Closed Systems"
Barcelona September 7-9 2011

Old questions and new contributions

How do quantum mechanics and statistical mechanics go together?

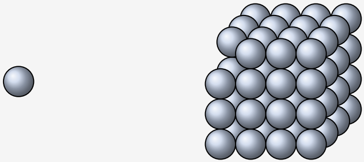
Many-Body Quantum Dynamics in Closed Systems



[1] M. Cramer, C. Dawson, J. Eisert, and T. Osborne, PRL 100 (2008) 030602
[2] P. Reimann, PRL 101 (2008) 190403

Many-Body Quantum Dynamics in Closed Systems

$\mathscr{H}_S \otimes \mathbb{1}$ $\mathbb{1} \otimes \mathscr{H}_B$



System "Bath"

[1] M. Cramer, C. Dawson, J. Eisert, and T. Osborne, PRL 100 (2008) 030602
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Many-Body Quantum Dynamics in Closed Systems

$$\mathscr{H} = \mathscr{H}_S \otimes \mathbb{1} + \mathscr{H}_{SB} + \mathbb{1} \otimes \mathscr{H}_B$$

$$|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}\mathscr{H}t}|\psi_0\rangle$$

$$A_t = \mathrm{Tr}[A|\psi_t\rangle\langle\psi_t|]$$

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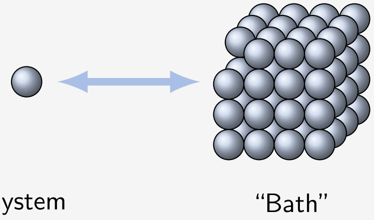

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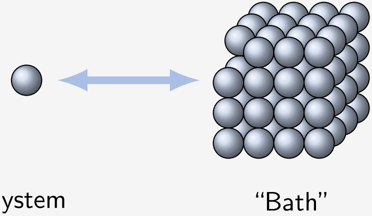

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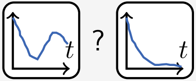
- Equilibration:



strong: equilibrated between t_1 and t_2 [1]

weak: equilibrated for most times [2]

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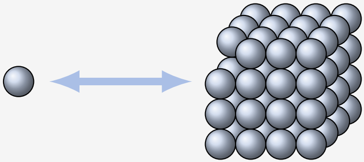

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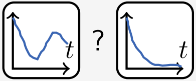

strong: equilibrated between t_1 and t_2 [1]

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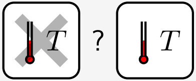
- Thermalization:



$$\psi_t^S \approx \rho_{\mathrm{Gibbs}} \propto \mathrm{e}^{-\beta \mathscr{H}_S}$$

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Equilibration and a maximum entropy principle

Maximum entropy principle

Theorem 1 (Maximum entropy principle [3])

If $\mathrm{Tr}[A\,\psi_t]$ *equilibrates, it equilibrates towards its time average*

$$\overline{\mathrm{Tr}[A\,\psi_t]} = \mathrm{Tr}[A\,\overline{\psi_t}] = \mathrm{Tr}[A\,\omega],$$

$$\textit{where } \omega = \sum_k \pi_k \psi_0 \pi_k$$

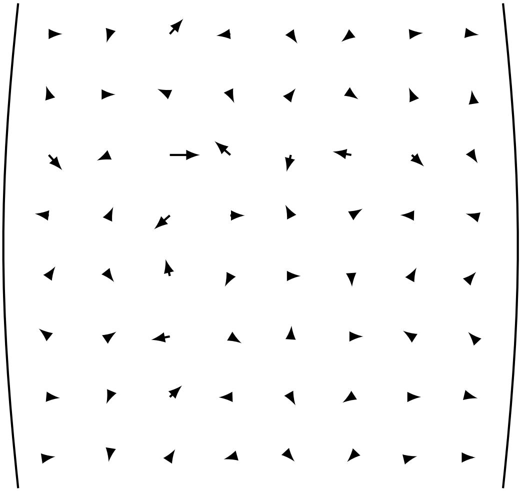
(with π_k *the energy eigen projectors) is the dephased state that maximizes the von Neumann entropy, given all conserved quantities.*

[3] C. Gogolin, M. P. Mueller, and J. Eisert, PRL 106 (2011) 040401

Maximu

Time averaging

$\psi_0 =$



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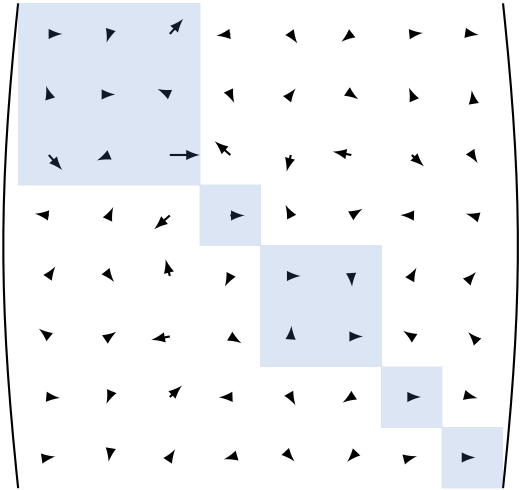

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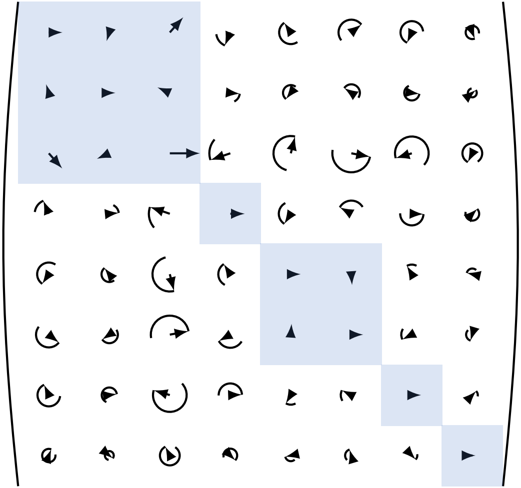
$\psi_t =$ 

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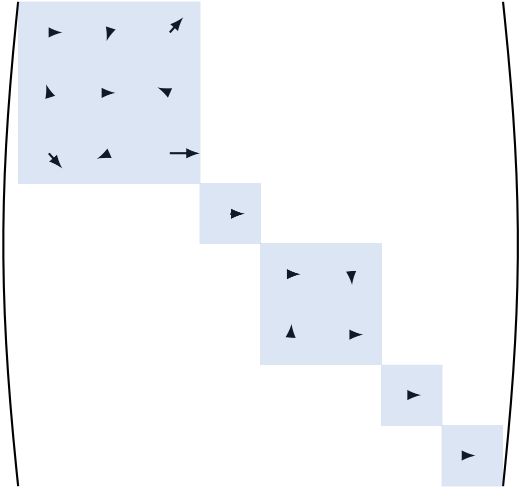

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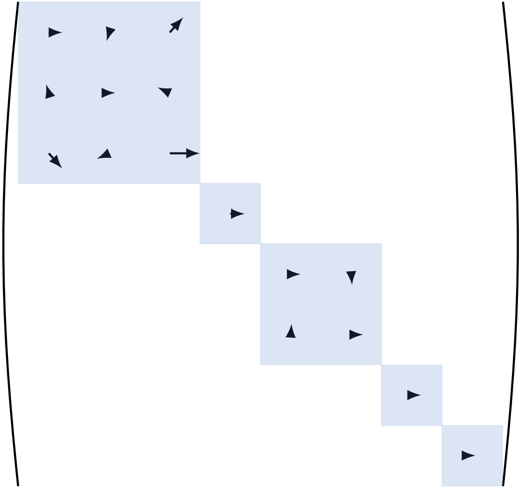
$\omega =$ 

$\psi_0 \to \omega$ is a pinching $\Rightarrow$ ω maximizes entropy.

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$\Rightarrow$ Maximum entropy principle from pure quantum dynamics.

Has nothing to do with (non)-integrability.

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Maximum entropy principle

Theorem 1 (Maximum entropy principle [3])

If $\mathrm{Tr}[A\,\psi_t]$ *equilibrates, it equilibrates towards its time average*

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Interesting open questions:

- Do we really need all (exponentially many) conserved quantities?
- If not, then which?
- Does this depend on integrability of the model?
- What is the relation to the GGE?

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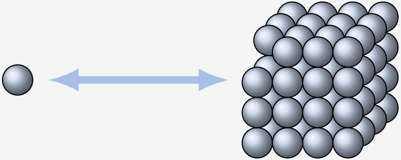
Thermalization and integrability

Thermalization is a complicated process



Thermalization implies:

1. Equilibration [2, 4, 5]
2. Subsystem initial state independence [3]
3. Weak bath state dependence [6]
4. Diagonal form of the subsystem equilibrium state [7]
5. Gibbs state $\mathrm{e}^{-\beta \mathscr{H}}$ [5, 6]

[2] P. Reimann, PRL 101 (2008) 190403
[4] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103
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[3] C. Gogolin, M. P. Mueller, and J. Eisert, PRL 106 (2011) 040401
[6] A. Riera, C. Gogolin, and J. Eisert, 1102.2389
[7] C. Gogolin, PRE 81 (2010) no. 5, 051127

Thermalization and quantum integrability

There is a common belief in the literature [8, 9, 10, 11, 12] . . .

Non-integrable	$\Longrightarrow$	Thermalization
Integrable	$\Longrightarrow$	No thermalization

[8] C. Kollath et. al PRL 98, (2007) 180601
[9] S. Manmana, S. Wessel, R. Noack, and A. Muramatsu, ibid. 98 (2007) 210405
[10] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452 (2008) 854
[11] M. C. Banuls, J. I. Cirac, and M. B. Hastings, arXiv:1007.3957
[12] M. Rigol, PRL 103, (2009) 100403

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Notions of (non-)integrability

A system is with n degrees of freedom is integrable if:

- There exist n (local) conserved mutually commuting linearly independent operators.
- There exist n (local) conserved mutually commuting algebraically independent operators.
- The system is integrable by the Bethe ansatz.
- The system exhibits nondiffractive scattering.
- The quantum many-body system is exactly solvable in any way.
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Lack of imagination?

Reminder on integrability in classical mechanics

Classical Liouville integrability

A system with n degrees of freedom is called integrable if it entails a maximal set of n independent Poisson commuting constants of motion and is called non-integrable otherwise [13].

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Classical:

- integrability $\Rightarrow$ systematic solvable and evolution on a n-torus

Quantum:

- always systematic solvable and evolution on a d-torus

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- thermalization $\Rightarrow$ non-integrability
  thermalization $\not\Leftarrow$ non-integrability

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Absence of thermalization in non integrable systems

Result (Theorem 1 and 2 in [3]):

- Too little (geometric) entanglement in the energy eigenbasis prevents initial state independence.
- This can happen even in non-integrable systems.

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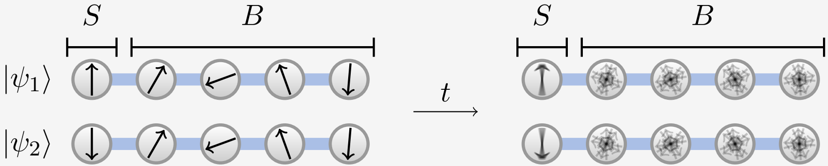

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Absence of thermalization in non integrable systems

The model:

Spin-$1/2$ XYZ chain with random coupling and on-site field.

$$\mathscr{H} = \sum_{i=1}^{n} h_i \, \sigma_i^Z + \sum_{i=1}^{n-1} \vec{b}_i \cdot \vec{\sigma}_i^{\mathrm{NN}}$$

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Interesting open questions:

- What is the relation to Anderson localization?
- Can this also happen in translation invariant systems?

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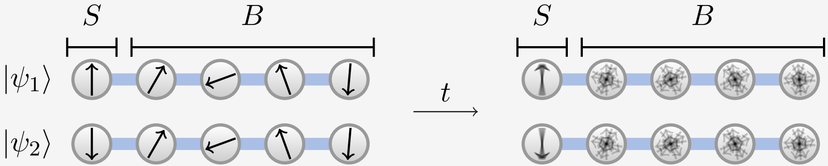

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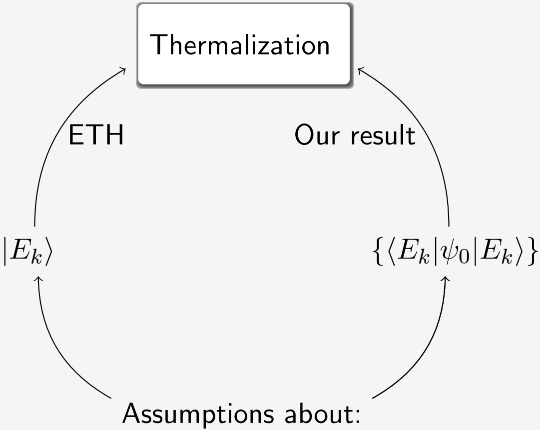
Proving thermalization

Two ways to prove thermalization



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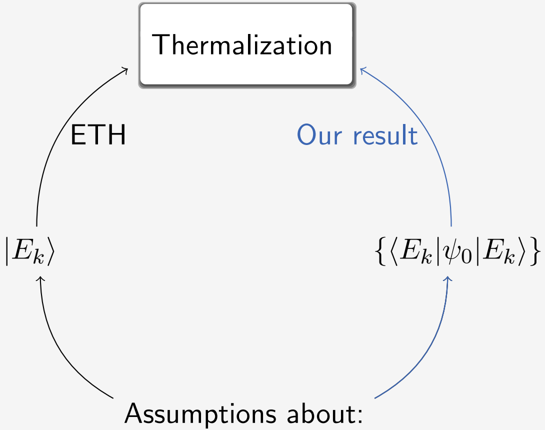

Structure of the argument

[14] S. Goldstein, PRL 96 (2006) no. 5, 050403
[6] A. Riera, C. Gogolin, and J. Eisert, 1102.2389

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Classical level counting à la Goldstein [14] with no interaction
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The result

$$\| \mathscr{H}_{SB} \|_\infty \gg \text{gaps}(\mathscr{H}_0)$$
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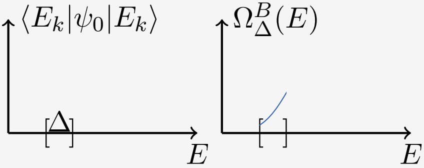

$\Longrightarrow$ "Theorem" 2 (Theorem 2 in [6])

(Kinematic) Almost all pure states from a microcanonical subspace $[E, E + \Delta]$ are locally close to a Gibbs state.

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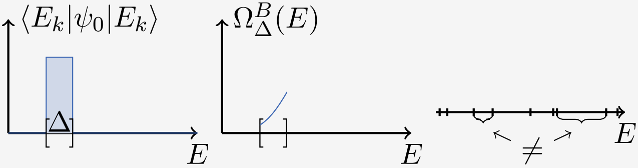

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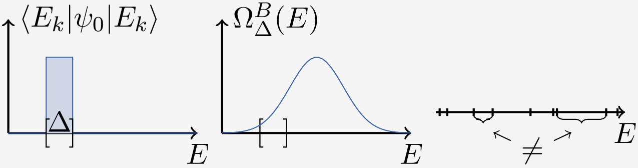

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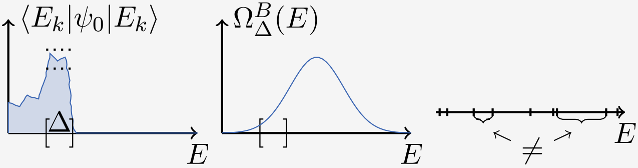

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- How are non-integrability and thermalization related?

Collaborators

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Arnau Riera

Martin Kliesch

Jens Eisert

PERIMETER INSTITUTE

Markus P. Müller

References

Thank you for your attention!

⟶ slides: www.cgogolin.de

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