A new foundation for Statistical Physics
Entaglement and the Second Law

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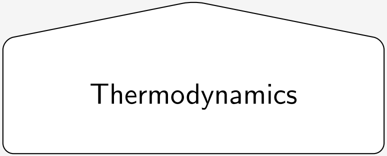
Canonical foundation of Thermodynamics

Thermodynamics

[1]

Canonical foundation of Thermodynamics



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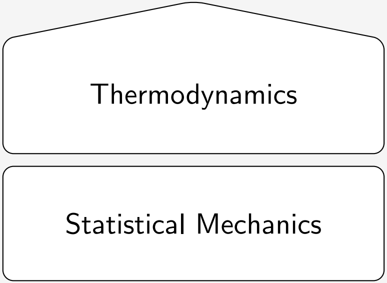

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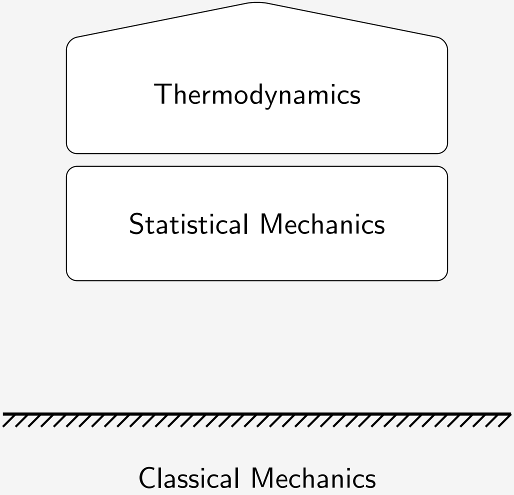

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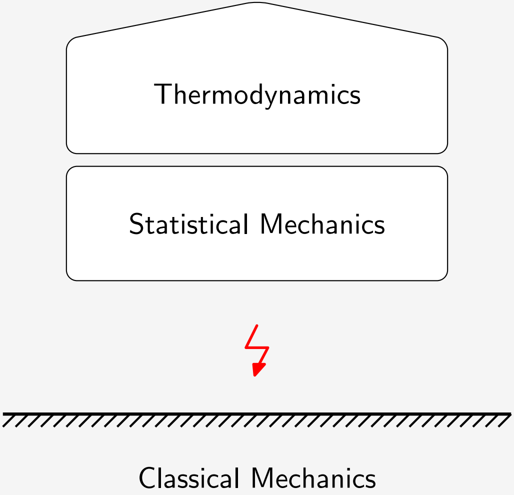

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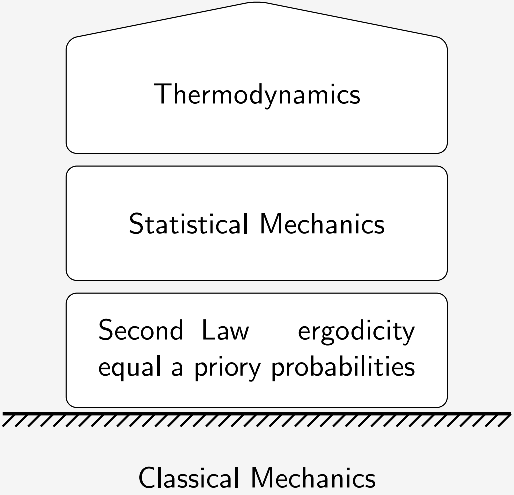

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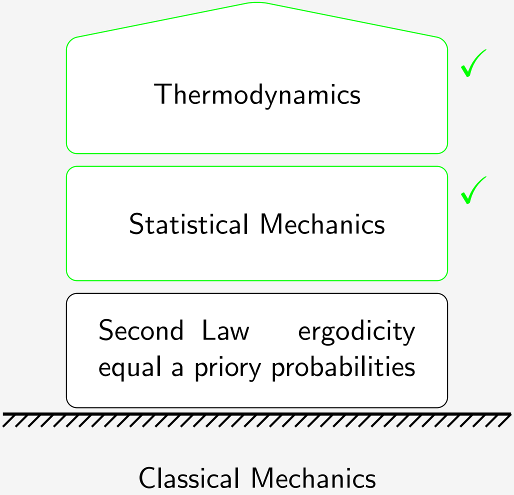

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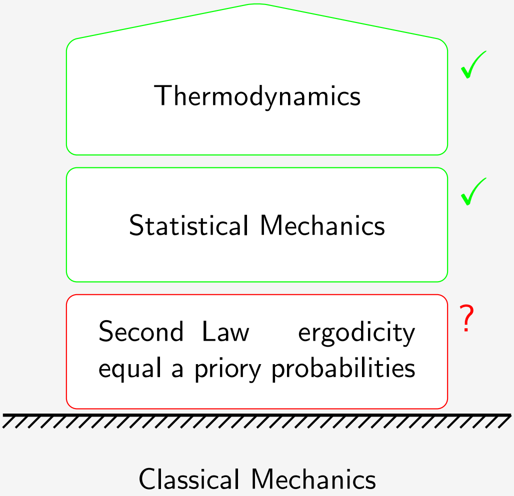

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What is wrong with the Second Law?

- Clausius

  Heat generally can not spontaneously flow from a material at lower temperature to a material at higher temperature.

[1,en.wikipedia.org]

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- Boltzmann (H-Theorem)

  The entropy in a closed system can not decrease. It stays constant only for reversible processes.

[1,en.wikipedia.org]

What is wrong with the Second Law?

H-Theorem:

Thus, one may prove that, because of the atomic movement in systems consisting of arbitrarily many material points, there always exists a quantity which, due to these atomic movements, cannot increase, and this quantity agrees, up to a constant factor, exactly with the value [of] the well-known integral $\int \frac{\delta Q}{T}$.

This provides an analytical proof of the Second Law [...] we immediately reach the result that $\int \frac{\delta Q}{T}$ is in general negative and zero only in a limit case.

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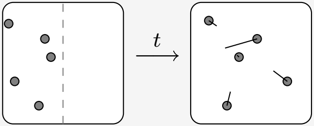
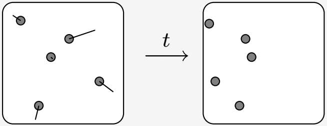
Time reversal objection (Loschmidt)



$\vec{v} \longrightarrow -\vec{v}$



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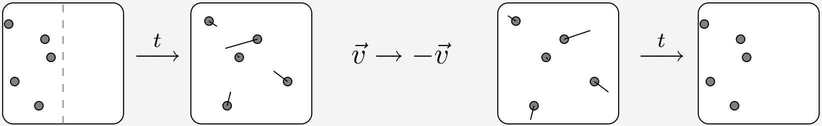
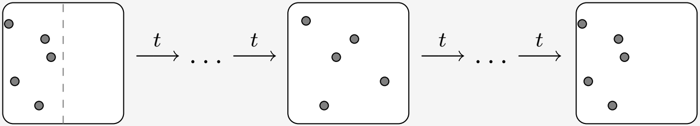


Recurrence objection (Ponicaré)



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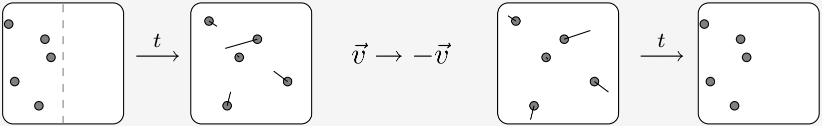

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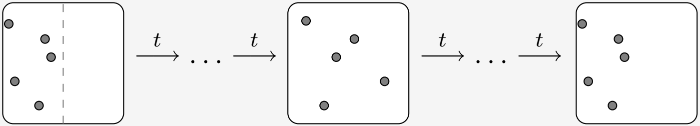

$\Longrightarrow$ We need a probabilistic H-Theorem! (Ehrenfest)

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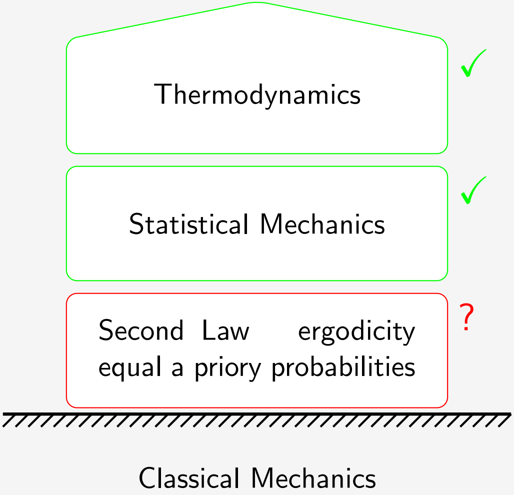
New foundation for Statistical Mechanics



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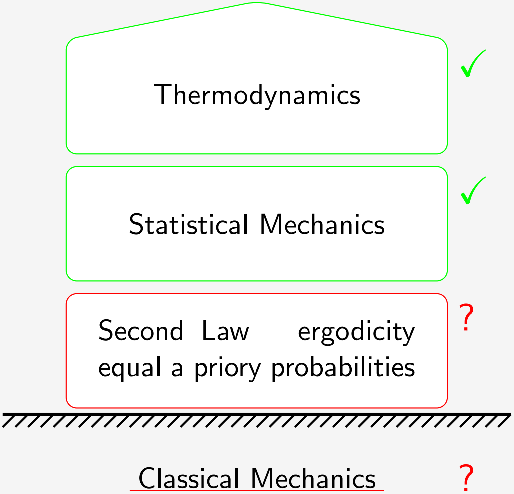

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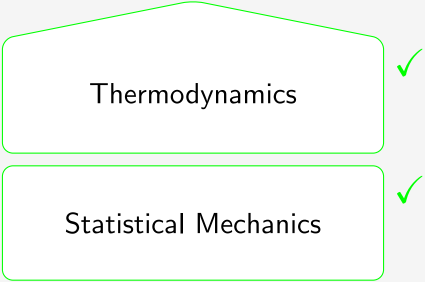

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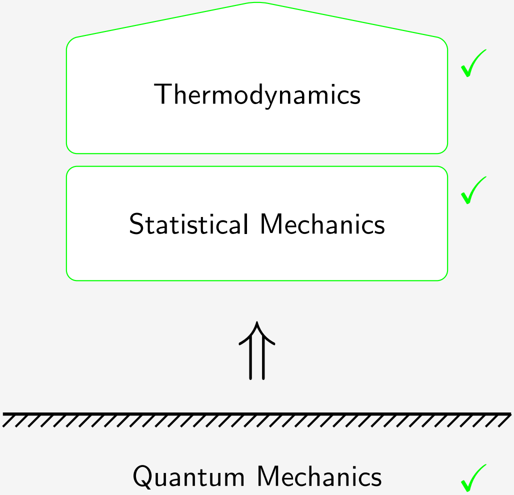

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Pure state quantum Statistical Mechanics ...

... must be capable of:

- reproducing results obtained from ensemble averages
- explaining equilibration
- explaining initial state independence
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Table of contents

Technical introduction

Quantum Mechanics on one slide

- Pure Quantum Mechanics

$$|\psi\rangle \in \mathcal{P}(\mathcal{H}) \qquad A = A^\dagger$$
$$\langle\psi|\psi\rangle = 1 \qquad \langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$
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$$\rho_t = U_t\,\rho_0\,U_t^\dagger \qquad U_t = \mathrm{e}^{-\mathrm{i}\,\mathscr{H}\,t}$$

mixtures: $\rho = p\,\psi_1 + (1-p)\,\psi_2$

Measures for distance and purity

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- Time average

$$\omega = \langle \rho_t \rangle_t = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \rho_t \,\mathrm{d}\,t$$

What is a random state?

- Parametrization

$$|\psi\rangle = \sum_i c_i |i\rangle \qquad \langle i|j\rangle = \delta_{ij}$$

$$1 = \sum_i |c_i|^2 \qquad \langle\psi|\psi\rangle = 1$$

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Haar measure

1. c_i normal distributed and normalize
2. Random U; $|\psi\rangle = U|0\rangle$

Setup

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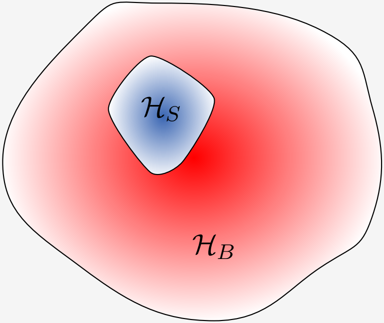
$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$$



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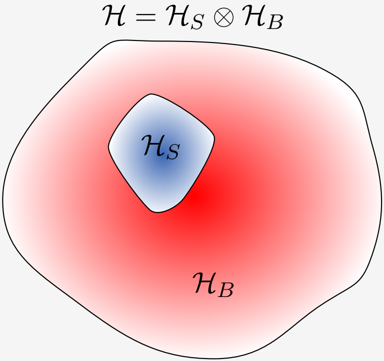

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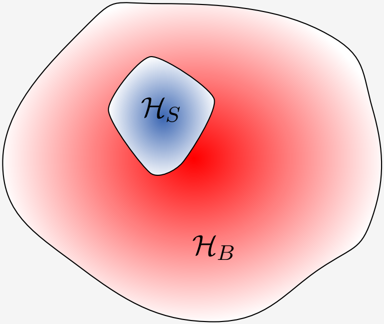

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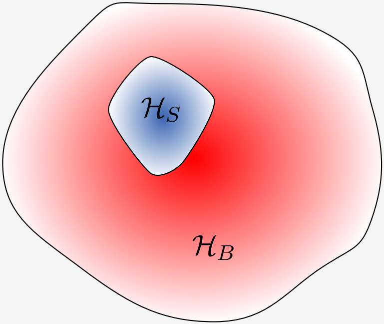

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reduced state → locally observable

A very weak assumption

In the following we assume non–degenerate energy gaps.

Definition

A Hamiltonian has non–degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\Longrightarrow k = l \wedge m = n \text{ or } k = m \wedge l = n$$

Equilibration

Two lemmas

Lemma 1

For every $\psi_0 \in \mathcal{P}(\mathcal{H})$

$$\langle \mathcal{D}(\rho_t^S, \omega^S) \rangle_t \leq \frac{1}{2} \sqrt{\frac{d_S^2}{d^{\mathrm{eff}}(\omega)}}$$

where

$$\rho_t^S = \mathrm{Tr}_B \, \psi_t \qquad \omega^S = \langle \rho_t^S \rangle_t \qquad \omega = \langle \psi_t \rangle_t$$

[4]

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$\implies \rho_t^S$ equilibrates if $d^{\text{eff}}(\omega)$ is large.

[4]

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Lemma 2

For random $\psi_0 \in \mathcal{P}(\mathcal{H})$

$$\Pr\left\{ d^{\text{eff}}(\omega) < \frac{d}{4} \right\} \leq 2\,\mathrm{e}^{-c\sqrt{d}}$$

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$\Longrightarrow d^{\mathrm{eff}}(\omega)$ is large if d is large.

[4]

Equilibration is generic

Lemma 1 + Lemma 2

If d is large ρ_t^S equilibrates for almost all initial states ψ_0.

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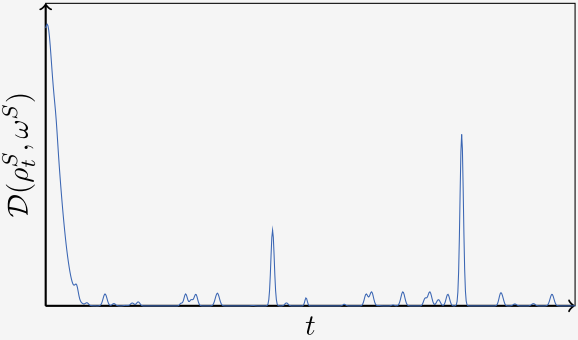

[4]

Initial state independence

Entanglement produces disorder

Lemma 3

If almost all energy eigenstates are close to locally completely mixed

$$\mathrm{Tr}_B\, |E_k\rangle\langle E_k| \approx \frac{1}{d_S} \mathbb{1}_{d_S \times d_S}$$

almost all equilibrium states are close to locally completely mixed

$$\omega^S = \langle \rho_t^S \rangle_t \approx \frac{1}{d_S} \mathbb{1}_{d_S \times d_S}.$$

Random Hamiltonians are locally mixed

Lemma 4

Almost all eigenstates $|E_k\rangle$ of a random Hamiltonian on $\mathcal{H}$ with $d_B \gg d_S$ are close to locally completely mixed in the sense that:

$$\Pr\left\{\forall k : \mathcal{D}(\mathrm{Tr}_B\,|E_k\rangle\langle E_k|, \frac{1}{d_S}\mathbb{1}_{d_S\times d_S}) \leq \frac{\epsilon}{d_S}\right\}$$
$$\geq 1 - 2\,d_S\,d_B\left(\frac{10\,d_S}{\epsilon}\right)^{2\,d_S} \mathrm{e}^{-C\,d_B\,\epsilon^2}$$

[5]

Toward a probabilistic H-Theorem

What have we learned so far?

Lemma 1 subsystems equilibrate $\Leftarrow$ effective dimension is high

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Lemma 4 random Hamiltonians have locally mixed energy eigenstates

What about the Second Law?

Probabilistic pseudo quantum H-Theorem

Almost all initial states of a large Quantum system of dimension d are such that under the time development induced by a generic Hamiltonian the states of all small subsystems with dimension $d_S \ll d$ are close to an equilibrium state for almost all times.
The equilibrium state is independent of the initial state and maximizes the local Von Neumann entropy.

Possible objections

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All I want to say is that:

> There is a generic tendency to maximize entropy.

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⟶ beamer slides: http://www.cgogolin.de

Thank you for your attention!